How do we represent geometry on a computer?

Enter: Descartes **Coordinates!**

(Let's pretend we can store arbitrary real numbers on a computer.)

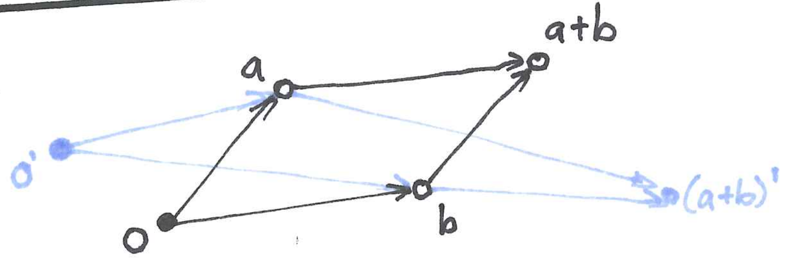
A point p in $\mathbb{R}^d$ has d real-valued coordinates. We write it as

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_d \end{bmatrix}$$

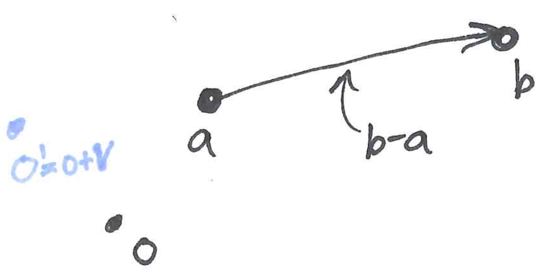
* points are not vectors
* points are like vectors

Basic Operations

	vectors	points
Addition	$u+v$	No
Subtraction	$u-v$	result is a vector
Scalar Mult	cv	No
dot product (inner)	$u \cdot v$, $\langle u,v \rangle$, $\boxed{u^T v}$	No
L_2-norm	$\|v\| := \sqrt{v^T v}$	No



Addition depends on the origin, (i.e. depends on coordinates)



$$(b-v)-(a-v) = b-v-a+v$$
$$= b-a.$$

Subtraction gives the "same" vector even if we move the origin.

Rotation?

Combinations

$v_1, \dots, v_n \in \mathbb{R}^d$

$\alpha_1, \dots, \alpha_n \in \mathbb{R}$

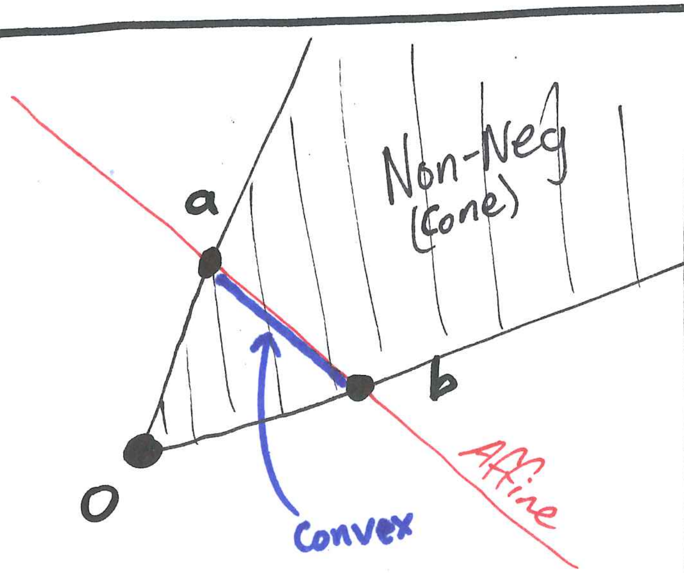
(1) Linear: $u = \sum_{i=1}^{n} \alpha_i v_i$

(2) Non-Negative: Linear and $\alpha_i \geq 0 \ \forall i$

(3) Affine: Linear and $\sum_{i=1}^{n} \alpha_i = 1$

(4) Convex: Affine and Non-Neg.

Work for points!



Affine

$p = \sum \alpha_i v_i \qquad \sum \alpha_i = 1$

$p' = \sum \alpha_i (v_i + u) = \left(\sum \alpha_i v_i\right) + \left(\sum \alpha_i u\right)$

$= p + u$

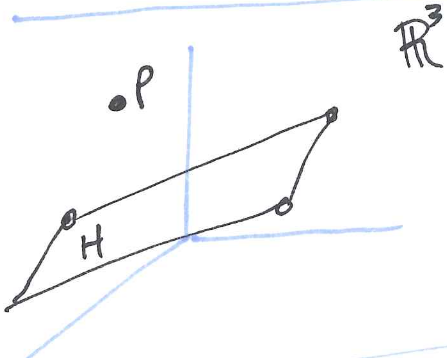
A Comparison Model for Geometric Algorithms

Recall $\Omega(n \log n)$ sorting L.B.
requires comparison $<, >, =$.
(No need to do arithmetic)

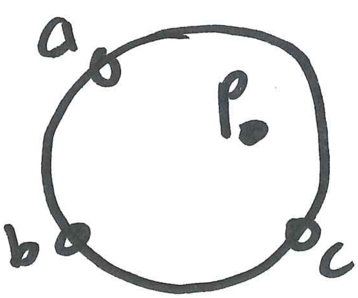
Linear Predicates

Is p above or below ℓ? or on

Is Δabc oriented clockwise or counter-clockwise?



Is p above or below hyperplane H?



Is p inside/outside/on the circumcircle of a,b,c?

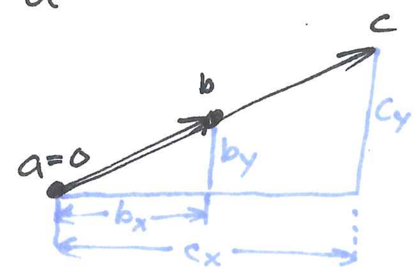
Lineside Test
(orientation)
(ccw)

Input: pts a, b, c

Output: $\begin{cases} +1 & \text{if } \Delta abc \text{ is ccw} \\ -1 & \text{if } \Delta abc \text{ is cw} \\ 0 & \text{if } a, b, c \text{ collinear} \end{cases}$

Easy Case:

Collinear when $a = 0$.



$$\left.\begin{matrix} b_x = \alpha c_x \\ b_y = \alpha c_y \end{matrix}\right\} \Rightarrow \frac{b_x}{c_x} = \frac{b_y}{c_y}$$

$$\Rightarrow b_x c_y - b_y c_x = 0$$

$$\Rightarrow \underbrace{\det \begin{bmatrix} b_x & c_x \\ b_y & c_y \end{bmatrix}}_{\det [b \;\; c]} = 0$$

Determinants of points?

It's okay, when $a \neq 0$

$$\det [b-a \quad c-a] = \det \begin{bmatrix} b_x - a_x & c_x - a_x \\ b_y - a_y & c_y - a_y \end{bmatrix}$$

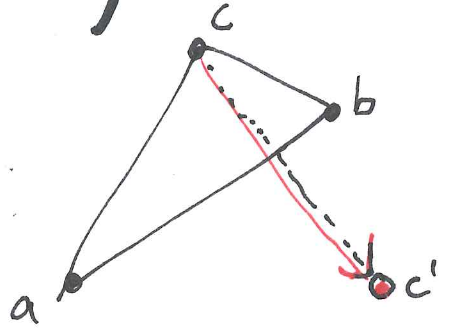
$$= (b_x - a_x)(c_y - a_y) - (c_x - a_x)(b_y - a_y)$$

$$= (b_x c_y - c_x b_y) - (a_x c_y - c_x a_y) + (a_x b_y - b_x a_y) \cancel{+ a_x a_y - a_x a_y}$$

$$= \det \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix}$$

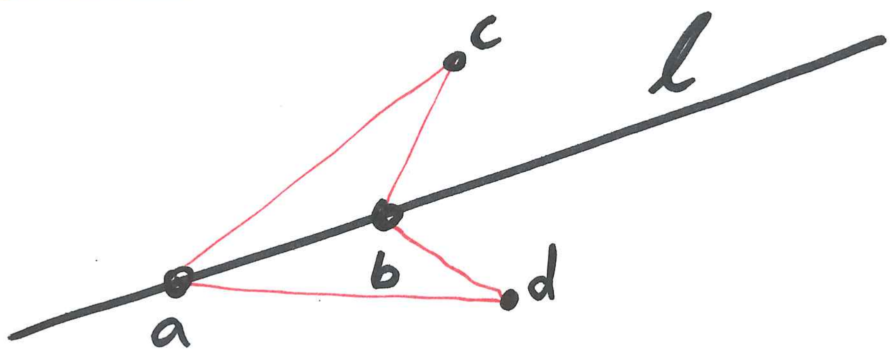
We thus discover homogenous coordinates

Recall that determinants give "signed" volume.

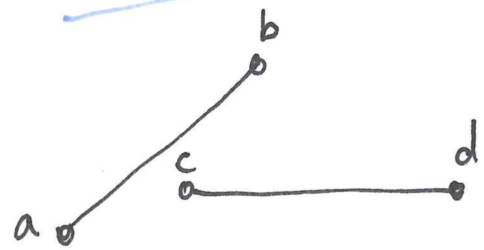


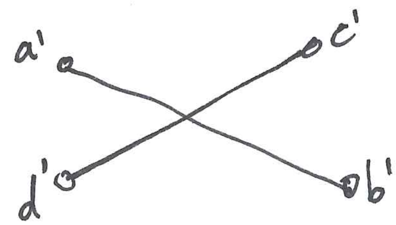
$$ccw(a,b,c) = \text{sign}\det\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}$$

3×3



We can now test if a point is above or below a line (given 2 pts on the line).





We can also test if two segments intersect.

<u>Claim</u>: $\overline{ab}$ intersects $\overline{cd}$ iff

$ccw(a,b,c) = -ccw(a,b,d)$ and

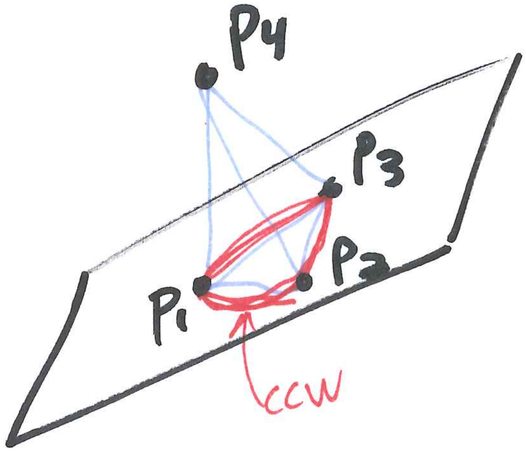
$ccw(c,d,a) = -ccw(c,d,b)$

assuming no 3 pts collinear.

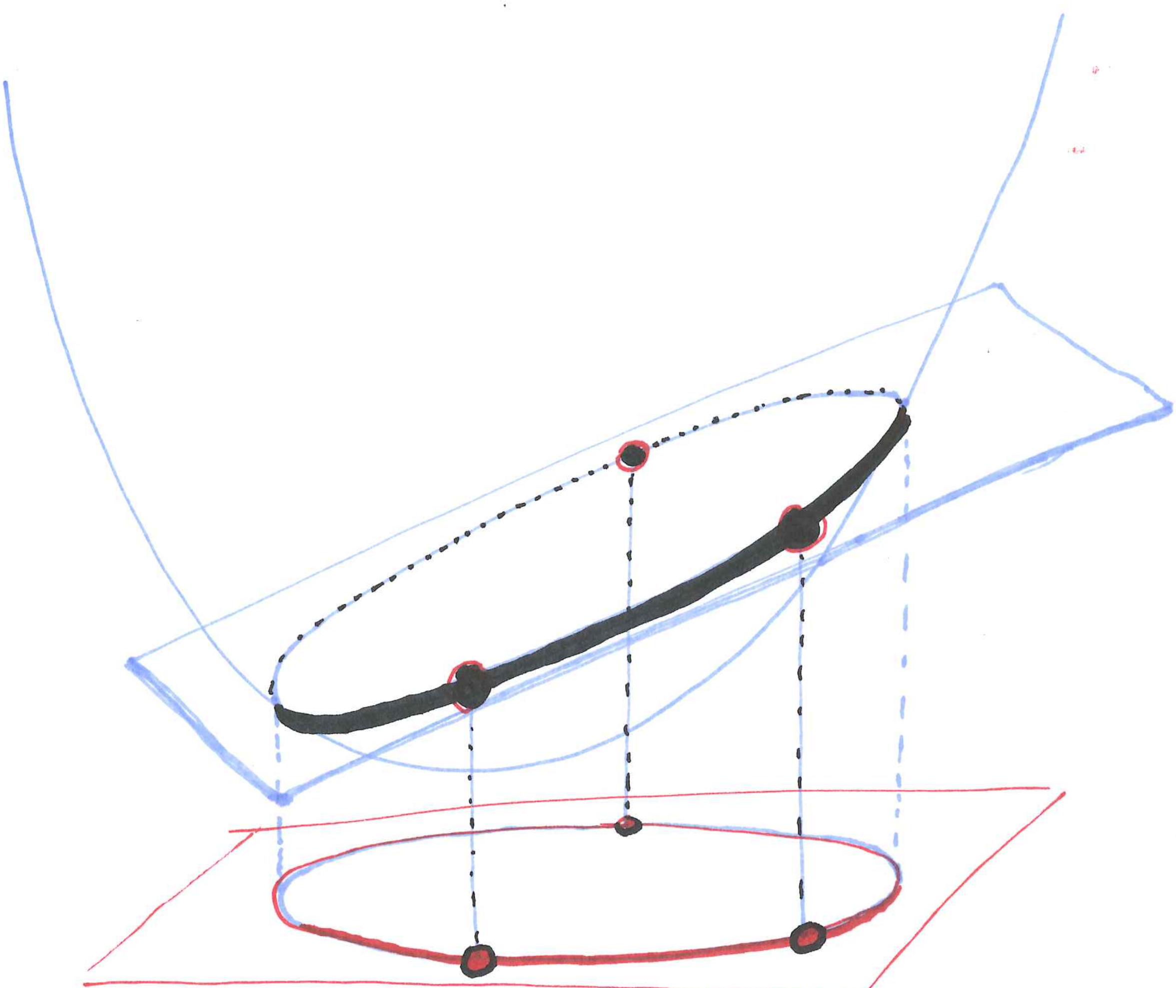
General Planeside Test
(orientation)

$p_1, \ldots, p_{d+1} \in \mathbb{R}^d$

$$\text{orient}(p_1, \ldots, p_{d+1}) := \text{sign}\left(\det\begin{bmatrix} p_1 & \cdots & p_{d+1} \\ 1 & \cdots & 1 \end{bmatrix}\right)$$



InCircle Test



Lift the points to a paraboloid $\Pi = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z = x^2 + y^2 \right\}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \longmapsto \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix} \in \mathbb{R}^3$$

Let H be the plane thru $\begin{bmatrix} a \\ \|a\|^2 \end{bmatrix}, \begin{bmatrix} b \\ \|b\|^2 \end{bmatrix}, \begin{bmatrix} c \\ \|c\|^2 \end{bmatrix}$

Let $v' = \begin{bmatrix} v \\ -\frac{1}{2} \end{bmatrix}$ be normal to H

So $H = \{ q' : q' \cdot v' = s \}$ for some s.

$s = v' \cdot \begin{bmatrix} a \\ \|a\|^2 \end{bmatrix} = v' \cdot \begin{bmatrix} b \\ \|b\|^2 \end{bmatrix} = v' \cdot \begin{bmatrix} c \\ \|c\|^2 \end{bmatrix}$

Suppose $p' = \begin{bmatrix} p \\ \|p\|^2 \end{bmatrix} \in H \cap \Pi$

then $\begin{bmatrix} p \\ \|p\|^2 \end{bmatrix} \cdot \begin{bmatrix} v \\ -\frac{1}{2} \end{bmatrix} = s$

$$\|$$

$$p \cdot v - \frac{1}{2}\|p\|^2 = s$$

$$2s = 2(p \cdot v) - \|p\|^2$$

$$\|p - v\| = \sqrt{\|p\|^2 - 2(p \cdot v) + \|v\|^2} = \underbrace{\sqrt{\|v\|^2 - 2s}}_{\text{indep. of } p}$$

Thus v is the circumcenter

and $\sqrt{\|v\|^2 - 2s}$ is the circumradius.