Everything we have done so far has led us to this class.

Lines + Circles 

↳ circumcircles

Predicates

InCircle Test 

Convex Hulls

PSLGs

Data Structures

Incidence Operations

(Updates)

Polyhedral Complexes

Triangulations of point sets

Delaunay Triangulation

Def For $P \subset \mathbb{R}^2$, Del_P is a triangulation of P s.t.

$\forall$ triangles $T \in P$, $\text{interior}(\text{circumcircle}(T)) \cap P = \emptyset$.

Equiv: $\Delta abc \in P$ and $d \in P \Rightarrow \text{InCircle}(a,b,c,d) \neq 1$

Note: this defn is in terms of a predicate.

Recall: InCircle Predicate

$$\text{InCircle}(a,b,c,d) = \frac{\text{sign}\left(\det\begin{pmatrix} a & b & c & d \\ \|a\|^2 & \|b\|^2 & \|c\|^2 & \|d\|^2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\right)}{\text{ccw}(a,c,b)}$$

It's a Linear predicate (planeside test) after lifting the points onto a paraboloid.

$$\underset{\text{pt in } \mathbb{R}^2}{x} \longmapsto \|x\|^2 = x_1^2 + x_2^2$$

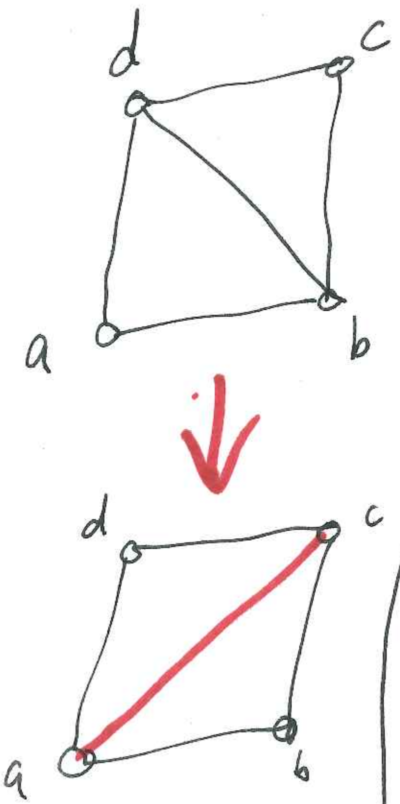
<u>Claim:</u>

$\triangle abc \in \text{Del}_P$ iff $\text{aff}\left(\begin{bmatrix} a \\ \|a\|^2 \end{bmatrix}, \begin{bmatrix} b \\ \|b\|^2 \end{bmatrix}, \begin{bmatrix} c \\ \|c\|^2 \end{bmatrix}\right)$ is a support plane for P.

This means that Del_P is the "Lower Hull" of P lifted to the Paraboloid.

Recall Also: Parabolic Lift for Sorting lower bound for convex hull problem... (same idea for 1D).

Flips



Given quad $\square abcd$ and edge $\overline{bd}$

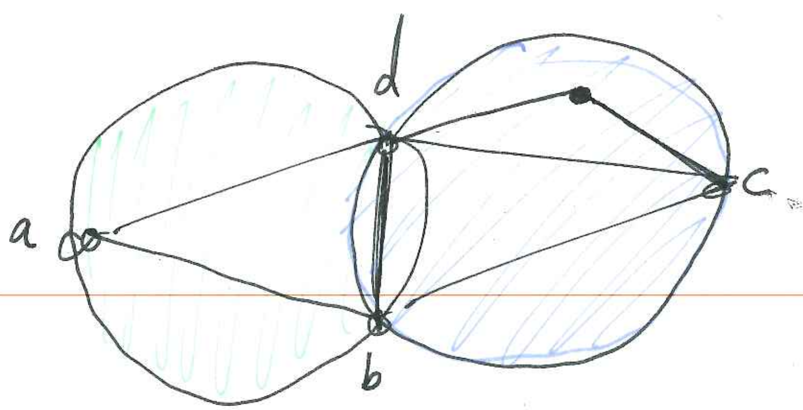
If $\square abcd$ is a convex quad, we can "flip" $\overline{bd}$, i.e. remove $\overline{bd}$ and insert $\overline{ac}$

We say $\overline{bd}$ is flippable if $\square abcd$ is a convex quad.

Def An edge $\overline{bd}$ is locally Delaunay (LD) if $\overline{bd} \in Del_{\{a,b,c,d\}}$

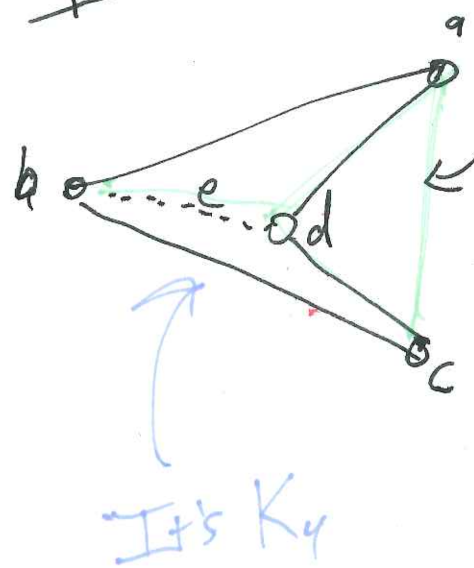
Equiv, $a \notin circle(bcd)$

$c \notin circle(abd)$



Claim e not flippable $\Rightarrow$ e is LD.

pf



$e = \overline{bd}$, $\square abcd$ non convex

$\Rightarrow$ one point inside Δ.

$Del_{\{a,b,c,d\}}$ has all possible edges.

So, e is LD.

The Contrapositive: If e is not LD then e is flippable.

An Algorithm:

GreedyFlipToDel (P)

Start with any Δ^n of P

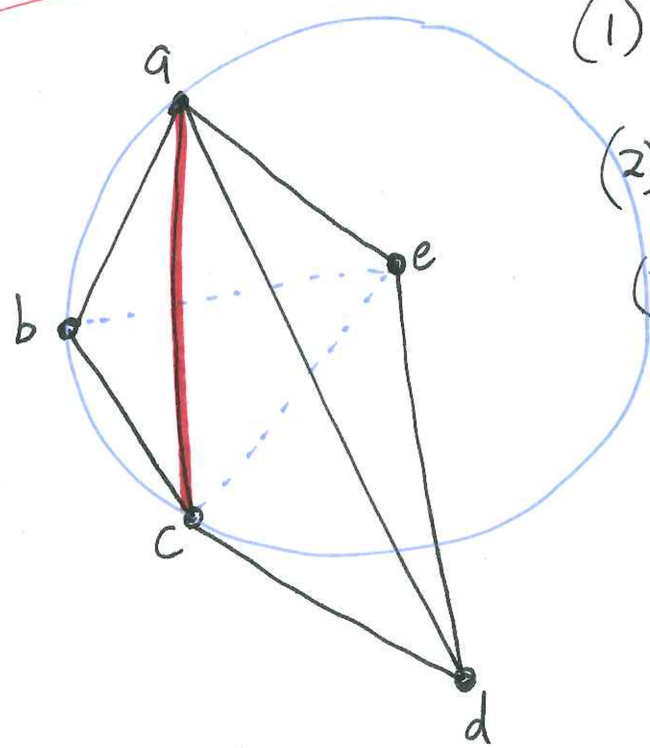
While $\exists e$ that is not LD

flip e.

Terminate? Delaunay Output?

A Closer Look at Locally Delaunay

LD $\not\Rightarrow$ Delaunay



(1) $\overline{ac}$ is LD but not Delaunay

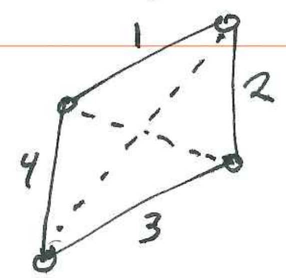
(2) $\overline{ad}$ is not LD

(3) $\overline{ac}$ is <u>not</u> LD after we flip $\overline{ad}$.

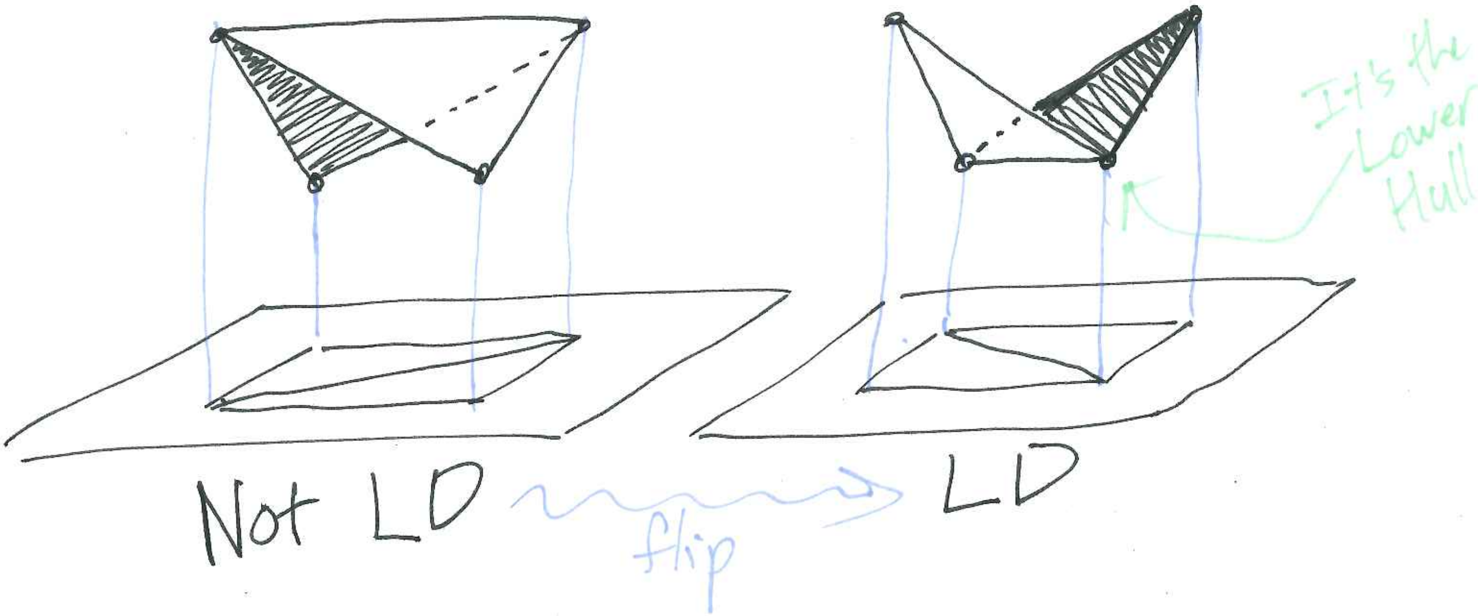
Local Conditions $\Rightarrow$ Constant time updates

Checking if an edge is LD takes one InCircle test.

At most 4 edges can change from LD to not LD.



Termination

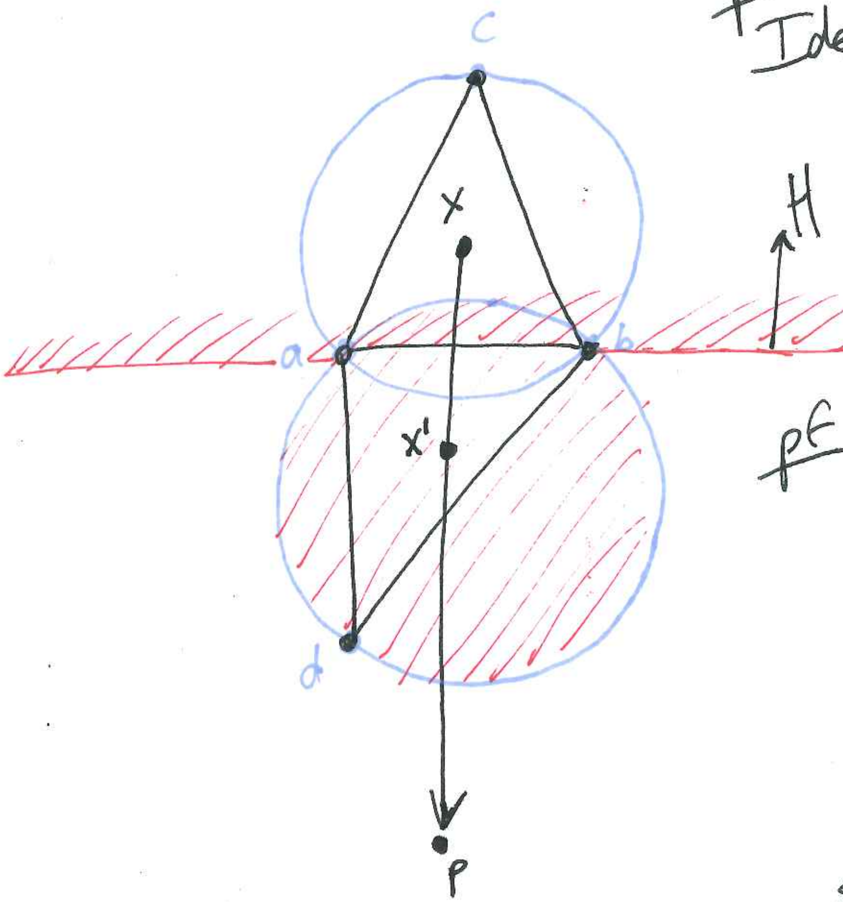


Each time we flip a non-LD edge, we decrease the volume below the "lifted triangulation."

Volume cannot go down forever. ⇒ Termination.

Count flips? Later.

<u>Thm</u> If T is a Δ^n of P s.t. all edges are LD, then $T = \mathrm{Del}_P$.



<u>pf sketch</u>

<u>Idea</u>: Pick any vertex and any Δ. Show the vertex is not in the interior of the circumcircle of the Δ.

<u>pf</u> Pick any $p \in P$.
Pick any $x \in CC(P)$
Let Δabc be Δ containing x.
We'll show $p \notin \mathrm{int}(\mathrm{circle}(abc))$.
By ind^n on $k = \#\{\text{edges crossed by } \overline{px}\}$

i.e. $p \notin H$

{WLOG, assume first edge is $\overline{ab}$.
Base $k=0$ is trivial.
Let d be vertex opposite c across $\overline{ab}$.
Pick $x' \in \overline{px} \cap \Delta abd$.
By ind^n, $p \notin \mathrm{circle}(abd)$.

$\overline{ab}$ is LD $\Longleftrightarrow$ $\boxed{\mathrm{circle}(abc) \subset H \cup \mathrm{circle}(abd).}$

$\Rightarrow p \notin \mathrm{circle}(abc)$.

Running Time

Counting Flips

Upper bound: $O(n^2)$

Why? Each edge is removed at most once.

For T a Δ^n, $h_T : CC(P) \to \mathbb{R}$ is the piecewise linear f^n we get by lifting vertices to the parabola and interpolating Δs.

If we flip $T \to T'$ then $h_{T'} \leq h_T$. (*)

If $x \in$ edge we flipped out, then

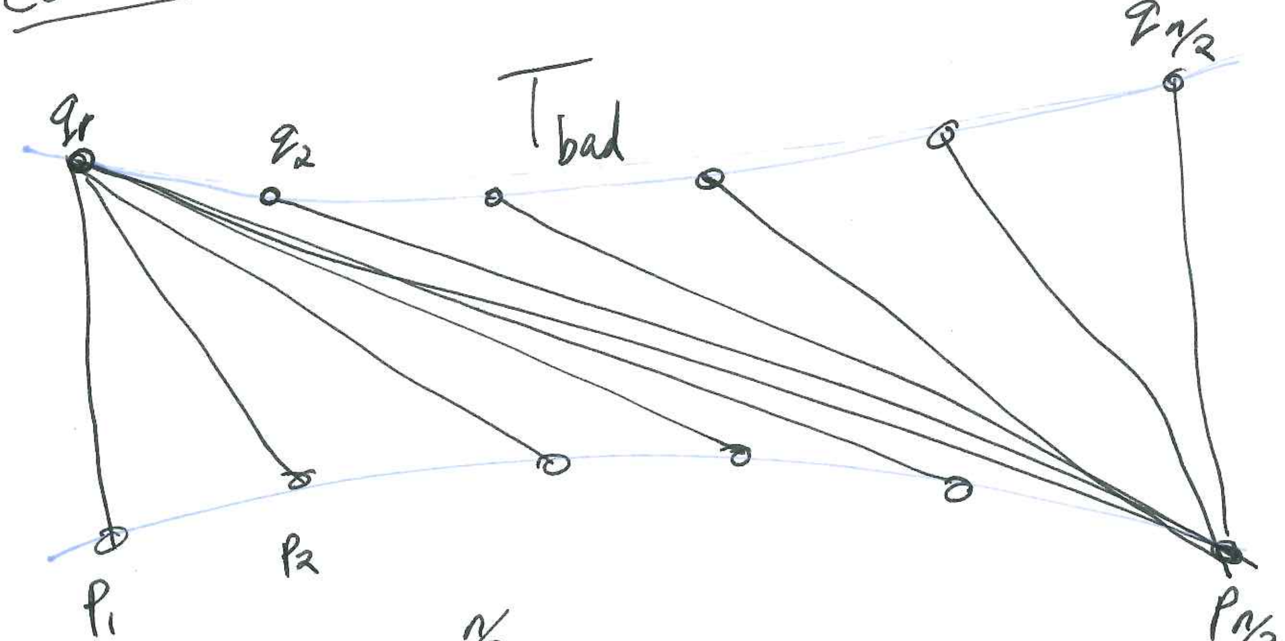
$$h_{T'}(x) < h_T(x).$$

If the edge appears later in T'' we have

$$h_{T''}(x) = h_T(x) > h_{T'}(x)$$

which contradicts (*).

Lower Bound



$$\Phi(T) = \sum_{i=1}^{n/2} \min_{q_j \sim p_i} |i-j|$$

$$\Phi(Del(P)) = 0$$

$$\Phi(T_{bad}) = \sum_{i=1}^{n/2} (i-1) = \Theta(n^2)$$

$T \to T'$ by one flip

$$\Rightarrow |\Phi(T) - \Phi(T')| \leq 1$$

$\Rightarrow$ Flipping T_{bad} to $Del P$ requires $\Theta(n^2)$ flips.

Wrapup

Flips $\rightsquigarrow$ Greedy Flip Algorithm.

Edges that are not LD can be flipped.

If we only flip non LD edge, we terminate.

If all edges are LD, we have ~~the~~ Del.

Running Time $O(n^2)$

Sometimes $O(n^2)$ flips are necessary.