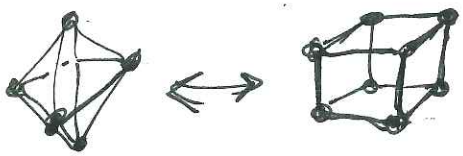
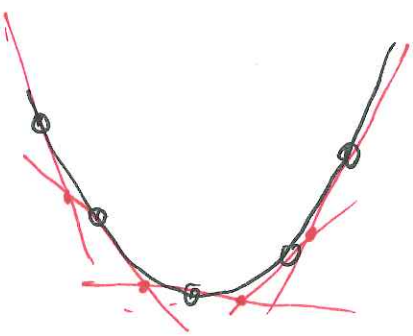
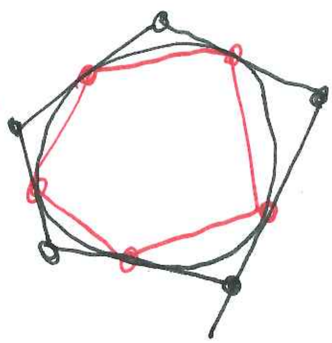
Today: Projective Duality
(AKA Polarity)



← These have both a geometric and a combinatorial relationship.

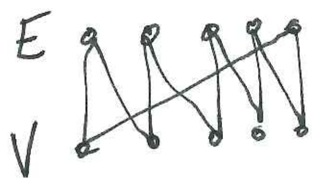
Let's start in the plane

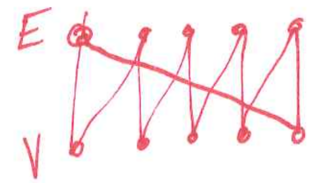




vertices → edges

edges → vertices



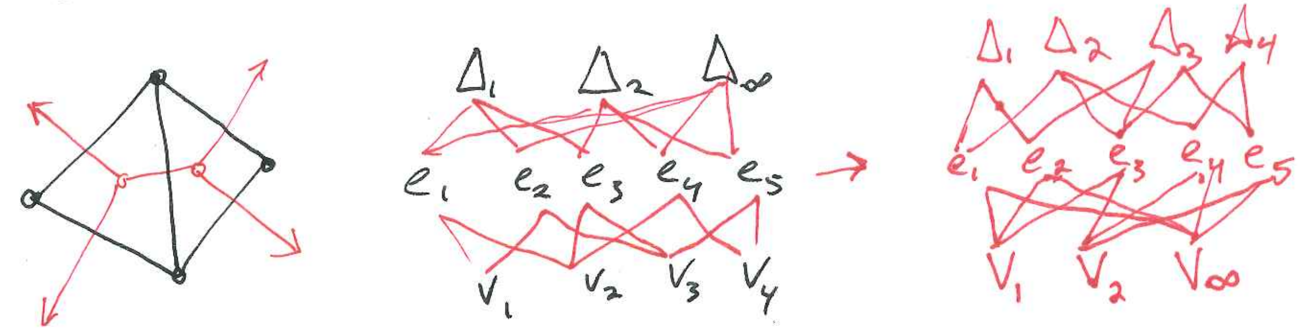


For points on a parabola, define their duals to be the tangent lines.

The Lower Hull $\xrightarrow[\text{(via duality}]{\text{becomes}}$ The Upper Envelope

This a type duality has a long history (but not that long in the scope of geometric history.)

Recall: Combinatorial Duality



We saw data structures that represent <u>both</u> the primal and the dual at the same time. (barycentric decomposition and (with a little work) half-edges)

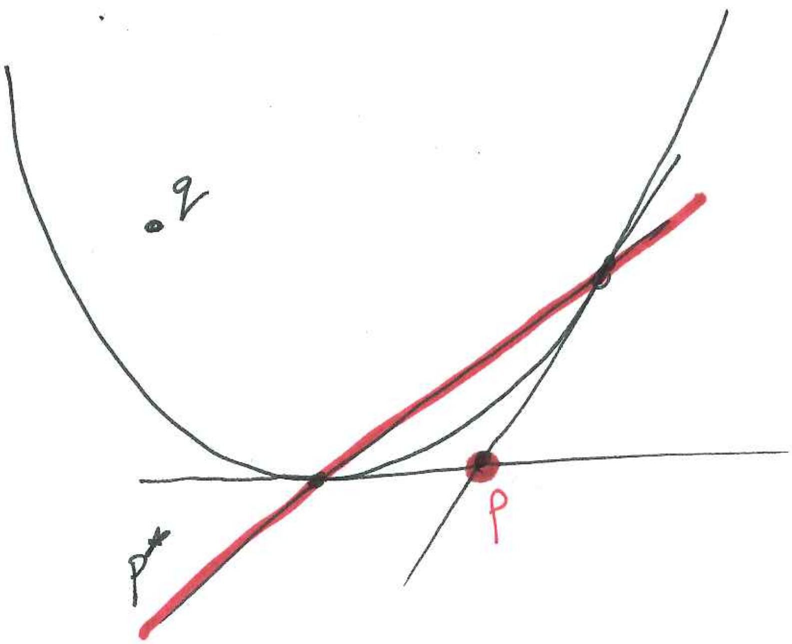
The duality implies we can use the same representation for both structures.

What else have we worked with that has this property? How about points and lines?

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \in \mathbb{R}^2 \qquad \ell = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = \ell_m x + \ell_b \right\}$$

we write $\ell = \begin{bmatrix} \ell_m \\ \ell_b \end{bmatrix} \in \mathbb{R}^2$

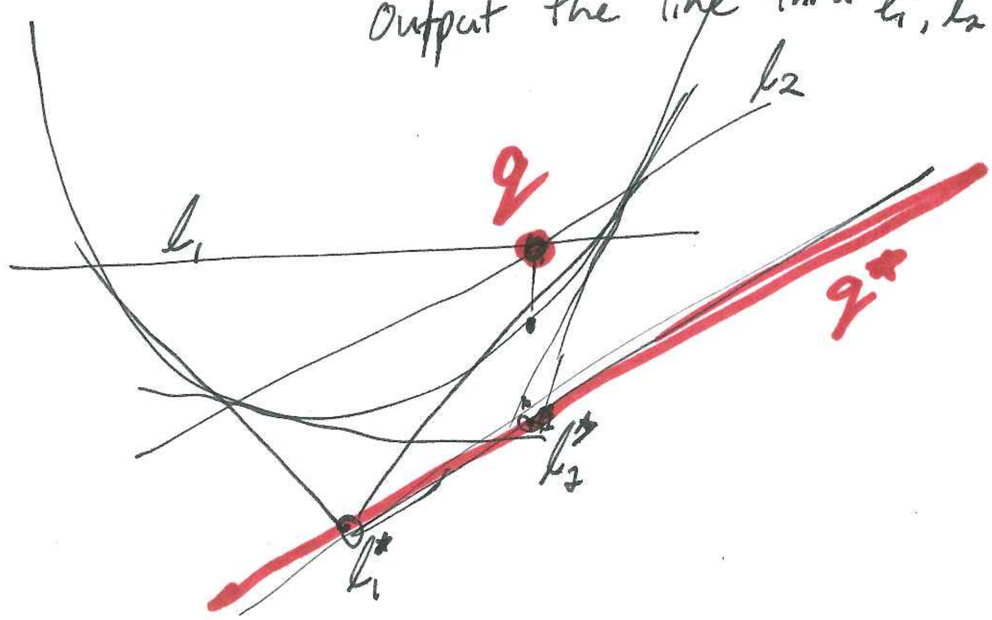
First, let's define duals for points <u>not</u> on the parabola.



Find 2 tangents to the parabola that pass thru p. Define the line p^* to be the line thru the 2 points of tangency.

What about q?

Take 2 lines thru q.
Take their dual points ℓ_1^*, ℓ_2
Output the line thru ℓ_1^*, ℓ_2^*.



How do we do this with coordinates?

p: point $\begin{bmatrix} p_x \\ p_y \end{bmatrix} \longleftrightarrow p^*$: line $\begin{bmatrix} 2p_x \\ -p_y \end{bmatrix}$

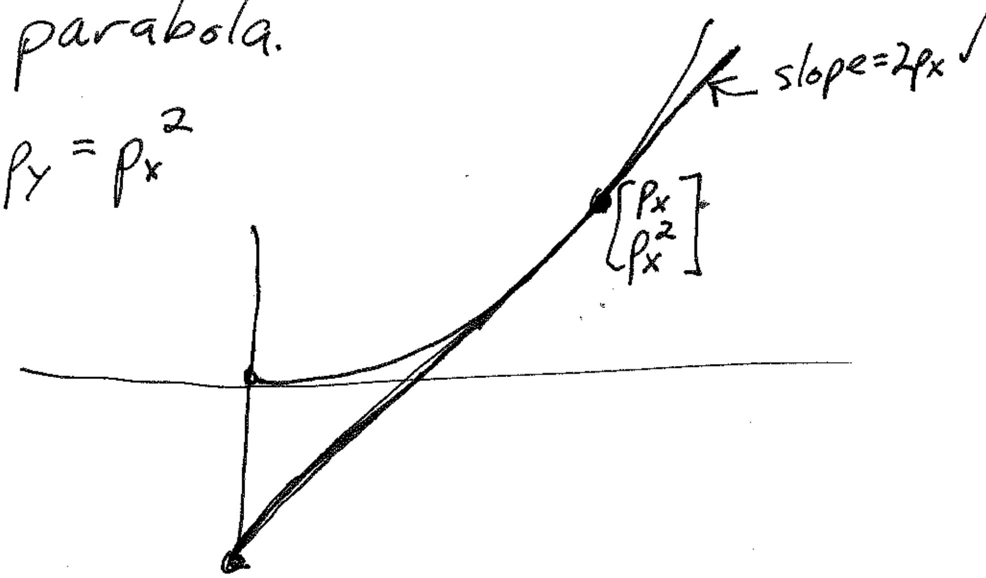
ℓ: line $\begin{bmatrix} \ell_m \\ \ell_b \end{bmatrix} \longleftrightarrow \ell^*$: point $\begin{bmatrix} \frac{1}{2}\ell_m \\ -\ell_b \end{bmatrix}$

First, check that $p^{**} = p$ and $\ell^{**} = \ell$.

This means we really have a "duality"!

Also, check that it does the "right" thing for points on the parabola.

$p \in$ parabola $\Rightarrow p_y = p_x^2$



Key Fact: Duality preserves incidence and above/below relation.

Claim: $p \overset{\text{above}}{\cancel{\in}} \ell$ iff $\ell^* \overset{\text{above}}{\cancel{\in}} p^*$

pf $\quad p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}, \; p^* = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = (2p_x)x - p_y \right\}$

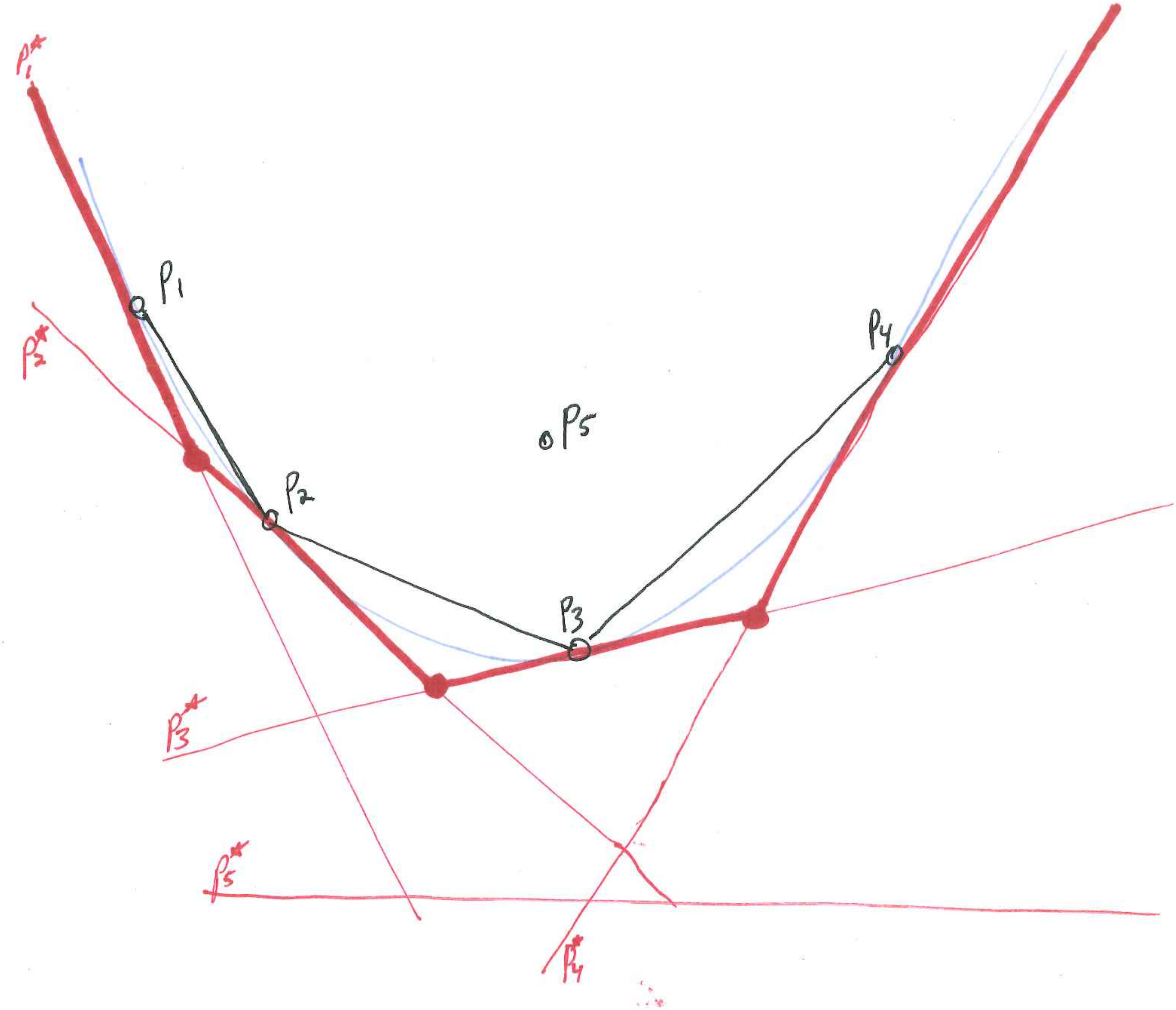
$$\ell = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = \ell_m x + \ell_b \right\}, \; \ell^* = \begin{bmatrix} \frac{1}{2}\ell_m \\ -\ell_b \end{bmatrix}$$

$$p \overset{\text{above}}{\cancel{\in}} \ell \iff p_y \overset{\geq}{=} \ell_m p_x + \ell_b$$

$$\iff -\ell_b \overset{\geq}{=} \ell_m p_x - p_y$$

$$\iff \underbrace{-\ell_b}_{\ell^*_y} \overset{\geq}{=} \underbrace{(2p_x)}_{p^*_m} \underbrace{\left(\tfrac{1}{2}\ell_m\right)}_{\ell^*_x} \underbrace{-p_y}_{p^*_b}$$

$$\iff \ell^* \overset{\text{above}}{\cancel{\in}} p^*$$



Lower Hull: $\overline{p_i p_j} \in LH$ iff $\forall k \notin \{i,j\}$. p_k is above $\overleftrightarrow{p_i p_j}$

$\Updownarrow$ Duality

Upper Envelope: $p_i^* \cap p_j^*$ is a vertex of the upper envelope iff $\forall k \notin \{i,j\}$ $p_i^* \cap p_j^*$ is above p_k^*

What about 3D?

A 3D point dualizes to a plane.

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad p^* = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = (2p_x)x + (2p_y)y - p_z \right\}$$

As in the plane the normal comes from the parabola.

Let $\bar{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$. The plane is $\left\{ z = 2\bar{p}^T \begin{bmatrix} x \\ y \end{bmatrix} - p_z \right\}$

<u>Recall</u>: Computing the Delaunay triangulation in $\mathbb{R}^2$ corresponds to computing the lower hull in $\mathbb{R}^3$. The lower hull dualizes to the upper envelope.

<u>Fact</u>: The projection of the upper envelope of the planes dual to a set of points $P \subset \mathbb{R}^2$ lifted onto the paraboid in $\mathbb{R}^3$ is exactly the Voronoi diagram. <u>In</u> this way, ~~th~~ we can see that the Voronoi/Delaunay duality is both combinatorial and geometric.