The Complexity of Domino Tiling Problems

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Abstract

In this paper, we give a combinatorial formalization that describes a wide range of related problems in the theory of tiling derived from the popular game of dominos. We show that determining if a given finite region of the plane can be tiled with a given set of dominos is NP-Complete. The new graph theoretical methods we introduce allow this result to be easily generalized to a variety of similar tiling problems. Several subproblems of the graph theoretical version of the problem are explored in depth. We then apply these methods to infinite tiling problems to lay the foundation for a more general study of undecidable tiling problems.

Contents

Chapter 1

Introduction

Problems of tiling spaces abound in discrete and computatial geometry. In this paper we explore a subclass of these problems inspired by the popular game of dominos. One recalls that a domino is a 2×1 rectangle with two square faces, each bearing some number of dots or pits. Although many variants to this basic domino exist and numerous games can be played with dominos, a commonality among these games is that when two domino tiles are placed adjacent to each other, the numbers on the corresponding faces should correspond. Some domino variants use colors on the tile faces rather than numbers. For the purpose of describing problems and results, it is clearer to refer to the color of the face so we will adopt this convention for the rest of this paper.

Tiling problems have been studied by computer scientists since the 1960's when Hao Wang [13] conjectured in 1961 that a set of 4-way dominos (now known universally as Wang tiles, see Figure 2.1) can tile the entire plane if and only if it can tile the plane periodically (i.e. in a pattern that repeates in two different directions). His corollary to this conjecture was that the decision problem of determining whether or not a given set of Wang tiles can tile the plane was decidable. In 1966, Berger [2] disproved the conjecture and showed how a Turing Machine could be simulated by a tiling of the plane. This result established the link between questions of computational complexity and tiling problems.

Also of interest to computer scientists are tiling problems involving the tiling of finite regions of the plane. The Jigsaw Puzzle Problem, the Monkey Puzzle problem and many other similar problems are known to be NP-Complete [8]. Also in the class of finite tiling problems are Polyomino problems, a rich and well-studied area of research (see [6]). Polyominos are tiles formed by gluing together unit squares edge to edge. For example, the 4-polyominos or tetrominos are familiar to many as the shapes from the video game Tetris. In general, polyominos do not have colored faces or the corresponding adjacency restriction so they come up only as a subproblem of the work presented here in which the number of colors is exactly one.

In this paper we start by looking at a specific problem of tiling finite regions with 2×1 dominos and settle the question posed by Watson and Worman [14] by proving the problem to be NP-Complete. First we show how to extend previous methods [14] to achieve this result, but then we shift our perspective to recast the problem combinatorially rather than geometrically. The main insight is that domino tiling problems contain two problems: a geometric problem, i.e. can a certain shape tile a certain space; and a combinatorial problem, i.e. can the tiles be rearranged to satisfy the adjacent color restriction. Work in polyominos has shown that the former is often NP-Complete. We show that even in cases where the geometric problem is tractable, the combinatorial problem remains NP-Complete. The focus on the combinatorial aspect of the problem over the geometric aspect marks a departure from the standard methods for proving hardness results for such problems.

In Chapter 2 we summarize relevant results from previous work done on related tiling problems. Chapter 3 explores a particular domino tiling problem and gives a graph theoretical generalization that extends results to other types of tiling problems. We consider the geometric problem as a subproblem of a purely combinatorial graph partitioning problem and derive NP-Completeness results independent of the geometry. Chapter 4 extends the methods of Chapter 3 to explore infinite tiling problems. We focus on the problem of modelling tiling problems with symmetry restrictions on the tiles. That is, problems in which the tiles can be flipped and rotated only according to certain rules. We also show how the local symmetry restrictions on the tiles is linked to the gloabal symmetry of the tilings that can be achieved and also to the decidability of the domino problem in its different manifestations. Finally, Chapter 5 is a summary of our conclusions along with discussion and some problems that remain open.

Chapter 2

Previous Work

As noted previously, much work in domino tiling problems started with Wang [13] and the Wang Tile (see Figure 2.1). Berger's proof of the falsity of Wang's conjecture implied the existence of sets of tiles that could tile the plane but could not be arranged to do so periodically [2]. Indeed, Berger showed that the problem of determining if a set of tiles yields a periodic tiling is undecidable. Berger demonstrated a set of tiles that yielded only aperiodic tilings of the plane. Unfortunately, his construction used 20,426 tiles. This result was simplified in 1971 by Robinson [12]. Culik [4] constructed an aperiodic set of only 13 Wang tiles. Note that these aperiodic tile sets are not undecidable because they have been proven to tile the plane. However, all undecidable tile sets will be aperiodic.

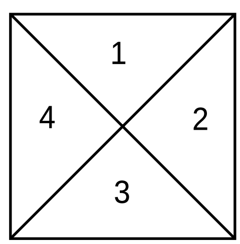

Figure 2.1: The standard Wang tile.

The problem of determining whether a set of Wang tiles can tile some finite region is known to be NP-Complete. In 2004, Watson and Worman [14] showed that determining if a set of dominos can tile a finite region is NP-Complete if an adversary is allowed to fix certain tiles in advance (the proof that the general version of this problem is NP-Complete is presented in Chapter 3).

In 1987 Grunbaum and Shephard [7] published *Tilings and Patterns* which surveys the entire scope of geometric tiling knowledge at that time and also presented numerous unpublished results. They note the lack of strong formal methods for attacking these problems. Since their writing several methods have been proposed. Radin [11] describes how aperiodic tilings can be described using Ergodic Theory. Conway introduced Algebraic techniques including the theory of tiling groups to give short certificates of non-tilability for polyomino problems (see [6] for a thorough survey of polyomino problems).

Recently tilings have found applications in the self-assembly of nanostructures (see [1]). Also, aperiodic Wang tilings are used to create nonrepeating images and texture maps for computer graphics (see [3]).

Remark. The term *Domino Tiling Problem* has been introduced several times in the literature to describe different problems. Wang used the term to describe what we now refer to as the Wang Tiling Problem. Watson and Worman use the term to describe the problem of tiling a space with the more common conception of a domino as a 2×1 rectangle. The term is also sometimes used to refer to the classical #P-Complete problem of counting the number of ways to tile a space with 2×1 rectangles that is well-studied for its relation to statistical mechanics.

Chapter 3

Finite Tiling Problems

3.1 Definitions and Formal Problem Statement

In the game of Dominos, the tiles in play are said to form the *layout*. Consequently, the term *layout* is sometimes used to denote the region to be tiled by the dominos. In some contexts, the term implies that the region to be tiled is the outline of some collection of dominos and the places where a domino can be placed within the region are prescribed. That is, there are only finitely many, nonoverlapping places where a domino can be placed (see Figure 3.1). In posing the Domino Tiling Problem, Watson and Worman [14] used this definition. In this paper, we will use the term layout to denote the region to be tiled but we abandon the added implication that the places where dominos can be placed within the layout is fixed in advance. Instead, we show that the hardness of the problem is independent of that condition and that the problem remains NP-Complete regardless of whether or not the layout contains information about the placement of dominos.

Ultimately, our goal will be to address these tiling problems as graph theoretical questions but many of our results will model geometric tilings so it is helpful to start with a geometric definition of a tiling in $\mathbb{R}^2$.

Definition 3.1.1. Geometric Deifnition of Tiling. A *tiling* of a region R is a decomposition of R into a countable number of subsets $S_1, \ldots, S_n$ where the interiors of the S_i's are disjoint and simply connected.

To rephrase this problem graph theoretically, it is first necessary to replace the plane $\mathbb{R}^2$ with a graph. We will require dominos to align edge to edge and vertex to vertex so we can imagine each domino occupying two adjacent cells of the integer

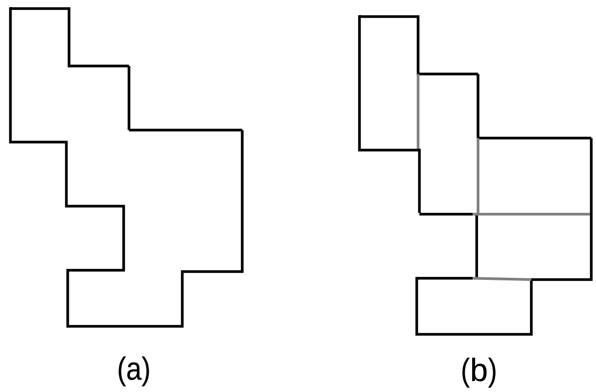

Figure 3.1: (a) A layout that is simply a region of the plane and (b) a layout that includes information about the placement of the tiles.

lattice $\mathbb{Z}^2$. So, we replace regions of the plane with subgraphs of the planar dual of the integer lattice. We will use the term layout and *layout graph* interchangeably to refer to this graph. We can now formally define the set of dominos and a tiling as graph theoretical notions.

Definition 3.1.2. The set of *dominos* D is a collection of isomorphic graphs with vertices assigned *colors* from a finite set C. We will denote the union of vertices (edges respectively) in all of the graphs in D by $V(D)$ ($E(D)$ respectively), and we denote the color of a vertex v by $c(v)$. We use the term n-domino to refer to a domino with n vertices.

We now see that the 2×1 dominos that we are dealing with would each be represented by a pair of vertices connected by a single edge in the domino set. If an edge in a graph is incident to two vertices of the same color, we say that it is *monochromatic*.

Definition 3.1.3. Graph Theoretical Definition of Tiling. A *tiling* of a layout graph G by a set of dominos D is a bijective map $T : V(D) \to V(G)$ that induces a surjective map $T' : E(D) \to E(G)$ in which all edges in $E(G) \setminus T'(E(D))$ are monochromatic. That is, if $T(u)T(v)$ is an edge in G but uv is not an edge in D then u and v must be the same color.

It is not hard to see that the formal definition above corresponds to the intuitive notion of how we expect a domino tiling to behave. Adjacent faces are either part of the same domino or they are the same color.

Problem 3.1.4. *The Domio Tiling Problem*
INSTANCE: A finite domino set D and a layout graph G.
QUESTION: Does there exist a tiling of G with D?

3.2 Standard Dominos in the Plane

A domino in the graph theoretical model of the Domino Tiling Problem is just a pair of vertices connected by a single edge. So, the tiling is just a perfect matching with an added restriction that edges not in the matching be monochromatic. Variants of this problems that define the layout to include the locations of domoinos effectively choose the perfect matching in advance. We will show that this distinction does nbot affect the hardness of the problem. The following simple lemma is useful.

Lemma 3.2.1. *Let the graph $G(V, E)$ be a tree. If G has a perfect matching M then M is the only perfect matching in G.*

Proof. We proceed by induction on $|V|$. Trivially, if $|V| = 2$ then there is only one edge and it constitutes the unique perfect matching on G. Assume the statement holds $|V| < k$. Suppose $|V| = k$ and G has a perfect matching M. Pick a leaf vertex v adjacent to a vertex u. The edge uv is the only edge adjacent to v so it is in M. By induction $M \setminus uv$ is the unique perfect matching on $G \setminus \{u, v\}$. So M is unique. □

We will show that the Domino Tiling Problem is NP-Complete even when the layout graph is a tree. Thus, Lemma 3.2.1 implies that it doesn't matter if we are given the matching in advance because there is at most one perfect matching and it can be found in polynomial time. If no perfect matching exists then trivially, there does not exist a tiling. If a color only appears on one vertex of one domino then that vertex must necessarily be mapped to a leaf vertex of the layout. This simple fact will be very helpful later so we state it formally below.

Lemma 3.2.2. *If there exists some vertex $v \in V(D)$ such that for all $u \neq v \in V(D), c(u) \neq c(v)$ then any tiling of a layout G with D maps v to a vertex of degree at most one.*

Watson and Worman [14] constructed a solver for 3,4-SAT out of dominos under the restriction that an adversary could fix certain tiles in advance. We prove that the reduction can still be performed without the adversary given only a polynomial increase in the size of the problem instance. Our purpose here is not to explain the exact workings of their construction but we will need to describe at least what it looks like in order to give an understanding of how to extend their methods and prove the general version of the problem they posed.

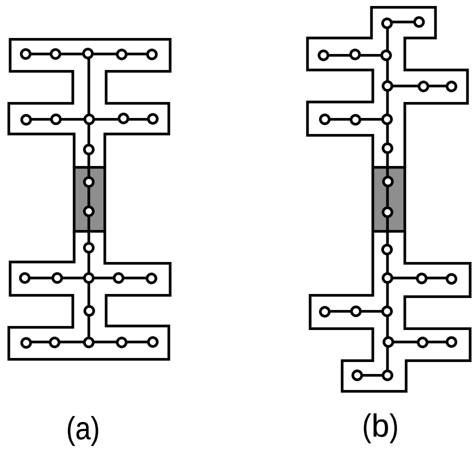

Figure 3.2: (a) The variable gizmo used in the Watson and Worman construction and (b) a modification that achieves maximum vertex degree 3.

The basic gizmo used in the Watson and Worman construction is shown in Figure 3.2.a. The gray tile is fixed by the adversary and represents a variable. Four clause gizmos, each a short path, are attached to the the four leaves above the fixed tile. The construction is such that all of the vertices above the fixed tile except for the one vertex adjacent to it must all have the same color in a valid tiling. The color corresponds to the truth assignment for the variable. The same holds for the vertices below the fixed tile. Consequently, we can modify the gizmo slightly as shown in Figure 3.2.b so that the maximum degree ov any vertex is 3.

Including clause and negation gizmos, the adversary fixes at most nine tiles for each variable gizmo. If the layout and domino sets are augmented so that a valid tiling exists if and only if the "fixed" tiles fall in the specified locations then the adversary can be eliminated and we get the following theorem.

Theorem 3.2.3. *The Domino Tiling Problem is NP-Complete for 2-dominos even when the layout graph is a tree with max vertex degree 3.*

Proof. As described above, it suffices to eliminate the adversary in the construction of Watson and Worman so that the layout has a tiling if and only if the pre-

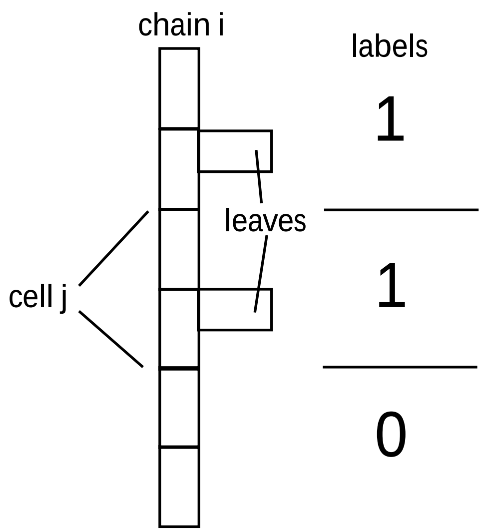

Figure 3.3: The anatomy of a chain.

scribed tiles fall in their prescribed locations. There are nine such tiles per variable gizmo. We add a chain of tiles adjacent to each “fixed” tile that uniquely identifies it. We construct the chains to have the following properties.

(1) The longest path in the chain has length $8\log(n)$ where n is the number of tiles to be fixed.

(2) No chain is isomorphic to any other chain.

(3) The longest path in the chain can be divided into $4\log(n)$ *cells*, each labelled 0 or 1 based on whether or not that cell contains a vertex of degree 3.

(4) The first $\log(n)$ and the last $\log(n)$ cells are labelled 0.

(5) The middle $2\log(n)$ cells encode a unique identifier for the tile to be fixed.

Figure 3.3 shows how such a chain is constructed. Note that $2\log(n)$ cells are needed to identify the tile to be fixed rather than just $\log(n)$ because then we would necessarily have chains that are isomorphic to other chains by reversing their order, violating property 2.

Add new colors to C for each pair (i, j) where i is the unique identifier for a chain and j is the cell number in that chain. We also add a unique color l_v for each leaf v in the chain.

We add dominos to the domino set in to fill in the chains. For each cell in a chain, add dominos colored $\{(i, j), (i, j-1)\}$ and $\{(i, j), (i, j+1)\}$. If that cell is labelled 1 then we also add a domino colored $\{(i, j), l_v\}$ where v is the leaf in the cell.

The colors added for leaves in the chains only appear on one vertex of one domino so by Lemma 3.2.2 that vertex must necessarily be mapped to a leaf vertex of the layout. In particular, that vertex color l_v will be mapped to the leaf v in the layout. This is true because property 2 and the fact that the color adjacent to the leaf appears on only two other dominos ensures that the rest of the chain is determined by the placement of a single leaf. Because the identifiers encoded by the chains are unique, no chain can be made to fit inside another. Also, no chain can be made to fit into a gizmo because the string of cells labelled 0 at either end of the chain is a path that is longer than any path that can appear in the gizmo.

All that remains is to add dominos to connect these chains to the dominos that they are designed to fix and connect all of the gizmos by another chain. Now, because the chains are determined and they terminate with the tile to be fixed, the adversary is unnecessary. This completes the proof.

□

3.3 Tiling Finite Graphs

3.3.1 Combinatorial Methods and Bigger Dominos

In this section, we examine a variety of subproblems of the graph theoretical version of the Domino Tiling Problem. Almost every nontrivial subproblem is shown to be NP-Complete. Some of the results given below could be obtained by constructions similar to those of the Theorem 3.2.3. However, we are able to present much simpler proofs by using the graph structure to our advantage and reducing from known NP-Complete problems in Combinatorics.

In Geometry, it is common to look at tiles ask questions about how they can be put together in a tiling with certain properties. In Graph Theory, it is more common to look at a graph and ask about how it can be taken apart (i.e partitioned) into smaller graphs with cetrtain properties. So, the graph theoretical version of the graph tiling problem that we present here could also be viewed as a graph partition problem. Kirkpatrick and Hell [9] showed that problem partitioning a graph into isomorphic subgraphs is NP-Complete when the subgraphs have at least 3 vertices. When the subgraphs have 2 vertices, the desired partition is a perect matching and can be found in polynomial time. Theorem 3.3.1 restates their result in the language of this paper.

Theorem 3.3.1. *The Domino Tiling Problem is NP-Complete for n-dominos for all $n \geq 3$ even when $|C| = 1$.*

Note that Theorem 3.2.3 and the results that follow show that the Domino Tiling Problem is NP-Complete even for graphs where the Partition into Isomorphic Subgraphs problem is tractable. The main theme of these results is that the combinatorial problem of satisfying the color constraints is sufficient to prove NP-Completeness.

The following theorem is a weaker result than that obtained in Theorem 3.2.3 but it has a much simpler proof that demonstrates the power of looking only at the graph representation.

Theorem 3.3.2. *The Domino tiling Problem is NP-Complete for 2-dominos even when the layout G is planar and has maximum vertex degree 5.*

Proof. The proof is by reduction from 3COLOR. Given a graph G that is an instance of 3COLOR. We construct G' an instance of the Domino Tiling Problem by adding a leaf to each vertex of G. We also replace every edge in G with a path of length 3. Figure 3.4 illustrates the modifications made to the original graph. We let $C = 1, 2, 3 \cup V(G)$ so that we have three colors plus a unique color for each of the original vertices. We construct the domino set D to satisfy the following rules.

(i) Every domino has a vertex colored in $\{1, 2, 3\}$,

(ii) the colors corresponding to $V(G)$ appear exactly once, and

(iii) no domino is monochromatic.

The only leaves in G' are the new vertices added so by Lemma 3.2.2 the leaves must all have colors in $V(G)$ for any valid tiling and thus each vertex from the original graph has a color in $\{1, 2, 3\}$. We can now see that a tiling of G' with D corresponds to a 3-coloring of G because if two vertices u and v were adjacent in G then they are separated by a path of length 3 in G'. That path is necessarily tiled

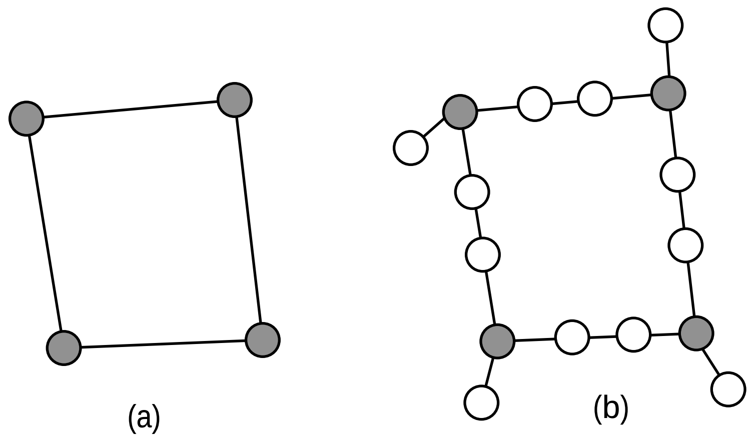

Figure 3.4: Modifying a graph for the reduction from 3COLOR.

by a single domino that is not monochromatic therefore u and v have different colors in the tiling. Conversely, if we are given a 3-coloring of G we can pick the domino set D to satisfy the given rules so that G' can be tiled by D. So, we have shown that G is 3-colorable if and only if there exists some set of dominos D satisfying the given rules such that G' can be tiled by D.

We complete the proof by showing that only a polynomial number of domino sets satisfy the given rules (ignoring duplicate sets which can produce identical colorings). There are $\binom{|V(G)|+2}{2}$ ways to assign the colors $\{1, 2, 3\}$ to the $|V(G)|$ dominos with one uniquely colored face (We do not need to distinguish between the unique colors). There remain $|E(G)|$ dominos to color, each only contains the colors $1, 2,$ and3 so there are only $\binom{|E(G)|+5}{5}$ choices for the coloring. Therefore, there are only a polynomial number of possible domino sets that satisfy the given rules and so, given an oracle for the Domino Tiling Problem we can try them all in polynomial time and get a polynomial time solution for 3COLOR.

3COLOR is known to be NP-Complete even for planar graphs of maximum vertex degree 4 (see [5]). Adding the leaf vertices and replacing edges with paths does not violate planarity and increases the degree of each vertex by at most one so the theorem follows. □

3.3.2 Unconnected Regions

In [14] a polynomial-time algorithm is given for the Domino Tiling Problem with 2-dominos in the case when the layout is a path or a cycle. The method used is quite elegant. The main idea is to construct the graph H where $V(H) = C$ and $E(H)$ is formed by adding an edge $c_i c_j$ for each edge $uv \in E(D)$ where $c(u) = c_i$and$c(v) = c_j$. The set D can tile a path (resp. cycle) if H is an Eulerian path (resp. cycle). This result highlights the fact that we can sometimes tell something about the hardness of a Domino Tiing Problem variant by looking at the structure of the layout graph. The following theorem shows that the nonconnectivity of this graph is sufficient to ensure the NP-Completeness of the corresponding problem.

Theorem 3.3.3. *The Domino Tiling Problem is NP-Complete even for 1-dominos when the layout G is not connected.*

Proof. We give a straightforward reduction from 3-PARTITION. Recall 3-PARTITION is a strong NP-Complete problem of the following form (the problem definition is from Garey and Johnson [5]).

Problem 3.3.4. *3-PARTITION*
INSTANCE: A finite set A of $3m$ elements, a bound $B \in \mathbb{Z}^+$, and a "size" $s(a) \in \mathbb{Z}^+$ for each $a \in A$, such that each $s(a)$ satisfies $\frac{B}{4} < s(a) < \frac{B}{2}$ and such that $\sum_{a \in A} s(a) = mB$.
QUESTION: Can A be partitioned into m disjoint sets $S_1, \ldots, S_m$ such that, for $1 \leq i \leq m$, $\sum_{a \in S_i} s(a) = B$? (Notice that the above constraints of the item sizes imply that every such S_i must contain exactly *three elements from A.)*

Suppose we are given an instance $\langle A, B, s \rangle$ of 3-PARTITION. Construct a layout graph G with $3m$ connected components of sizes $s(a)$ for all $a \in A$. Let D be a set of mB 1-dominos colored in $\{1, \ldots, m\}$ with exactly B dominos colored i for all $1 < i < m$.

Now, suppose there exists the subsets $S_1, \ldots, S_m$ of the 3-Partition. We can tile G' with D by a mapping all vertices colored i to the three connected components corresponding to the elements of S_i. Conversely, a valid tiling must have the property that all dominos mapped to the same connected component have the same color. So, it makes sense to think of the connected components as being colored from $\{1, \ldots, m\}$. Each connected component corresponds to an element of A so the coloring gives a partition of A into m subsets and because the number of dominos of any one color is B, the sum of the sizes of the of elements in the subsets is always B. Therefore, $\langle A, B, s \rangle \in$ 3-PARTITION if and only if G can be tiled by D.

Because 3-PARTITION is NP-Complete in the strong sense, we may assume that the numbers are expressed in unary. This is necessary to avoid an exponen-

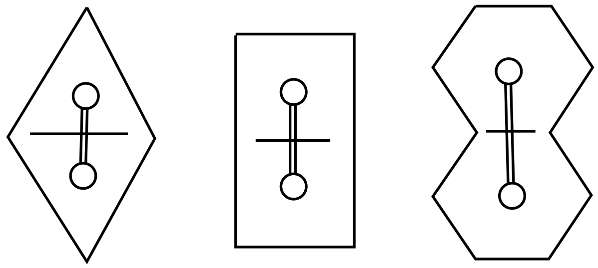

Figure 3.5: Different domino shapes for the plane.

tial blowup in the size of the problem instance when we convert the numbers to subgraphs with the corresponding number of vertices. □

If we restrict the number of colors, it is possible to solve this problem in polynomial time.

Theorem 3.3.5. *There exists a polynomial time solution for the Domino Tiling Problem with 1-dominos when* $|C| = 2$*.*

Proof. In polynomial-time we can traverse G and record the sizes $s_1, \ldots, s_k$ of connected components. Now, $|C| = 2$ so let r be the number of vertices in $V(D)$ of one color and let s be the number of vertices of the other color. The problem now is to find a subset of $\{s_1, \ldots, s_k\}$ that sum to r. This is simply the SUBSET-SUM problem which is NP-Complete but only in the weak sense. Therefore, we can solve the problem in time exponential in the size of the representation of $\{s_1, \ldots, s_k\}$ which is polynomial in the size of G. □

3.4 Restriction to the Plane

Section 3.3 treated the Domino Tiling Problem from a purely graph theoretical perspective with a slant towards analyzing subproblems that may be represent geometric tiling problems. One advantage of this approach is that our methods allow for simple generalization to other sort of dominos. Figure 3.5 depicts some domino variants for the plane that also yield NP-Complete Domino Tiling Problems. The triangular domino in the figure was the primary motivations for adding maximum vertex degree 3 restriction in Theorem 3.2.3. Most of the theorems presented thus far do hold even when the layout graph is planar but not all represent tiling problems in the plane.

3.5 A Generalized Notion of Tile Colors

The rules for adjacency in dominos is simple, if two faces have the same color then they can be placed edge to edge. So, the relation of being able to be placed adjacently is transitive: if face A can be placed adjacent to B and face B can be placed next to C then A and C can be placed adjacent to one another. Consider a common type of tiling, a jigsaw puzzle. In a jigsaw puzzle this transitivity does not usually hold. There are edges with protrusions and those with intrusions. We would like to generalize the notion of coloring capture the idea of nontransitive adjacency rules.

We can build a graph that models the valid adjacencies of a tiling. Define the *adjacency graph* $G_A(V, E)$ for a domino set D in the following way. The vertex set is the set of domino vertices and the edge set contains an edge between any two vertices that can be placed adjacent to each other in a tiling. So, we have $V = V(D)$ and $E = \{uv \,|\, u, v \in V(D)$ and u can be placed next to v in a valid tiling$\}$.

Observe that when we color the vertices and allow the vertices of the same color to be placed next to each other, the adjacency graph G_A is a union of cliques. To model a problem where the tiles fit together like jigsaw piece, we might color the vertices as before but also mark some vertices as intrusions and other as extrusions. In this case the adjacency graph is a collection of complete bipartite graphs. In particular, for the jigsaw puzzle where each piece matches exactly one other piece, the adjacency graph is a perfect matching.

Questions regarding tiling layouts with dominos where the adjacency graph is an arbitrary graph have not been explored. This new class of problems can be viewed as a partition of a layout graph G into subgraphs isomorphic to D under the constraints defined by G_A. One interesting problem that can be stated in this formalism is what we refer to as the Poorly Cut Jigsaw Puzzle. In this problem, tiles A and B might both fit a tile C but it might be that there is a tile D that only one of the two fits. This problem also could be said to correspond with the idea of allowing a hammer to force jigsaw puzzle pieces together that otherwise wouldn't fit. A variety of other "loose" tiling problems are also easy to formulate. These methods may have applications for addressing tiling approximation problems.

Chapter 4

Infinite Tiling Problems

4.1 Local and Global Symmetry

Formally, an infinite tiling is called periodic if it has two linearly independent translational symmetries. Intuitively, a tiling is periodic if it repeats in two different directions. We say that a set of tiles is aperiodic if they can tile the plane but can only do so aperiodically. It is not immediately obvious that such tile sets should exist.

Problem 4.1.1. *Wang Tile Problem*
INSTANCE: A finite, nonempty set S of Wang Tiles.
QUESTION: Does there exist a tiling of the plane using only tiles from S and disallowing rotations of the tiles?

As discussed in Chapter 2, the work of Hao Wang and Robert Berger in the 1960's laid the foundation for much work future work on the decidability of infinite tiling problems. Wang correctly identified the connection between the decidability and periodicity of the tiling, but he incorrectly conjectured that all infinite tilings could be made periodic and therefore were all decidable. Berger disproved the conjecture and proved the following.

Theorem 4.1.2. *(Berger) The Wang Tile Problem is Undecidable.*

It is important to note that restriction about the rotation of the tiles is essential. This fact is amply demonstrated in the following theorem.

Theorem 4.1.3. *The Wang Tile Problem is trivial if the tiles can be rotated.*

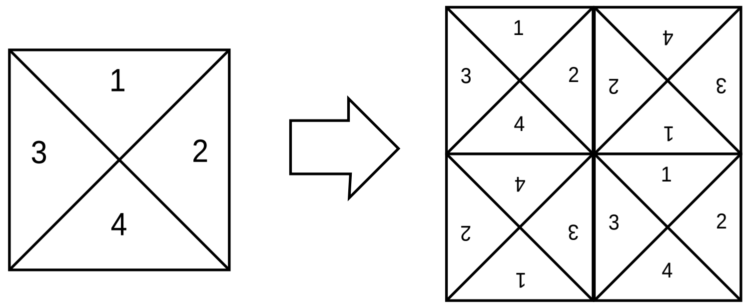

Figure 4.1: A single rotatable Wang tile can tile the plane.

Proof. Figure 4.1 shows how four copies of a single Wang tile can be rotated and tiled to form a larger tile. Note that the collection of tiles has identical right and left sides as well as identical top and bottom sides. So, copies of these four tiles can tile the entire plain. It follows that every instance of the Wang Tile Problem with rotatable tiles has a solution.

□

Recall that the undecidability of the Wang Tile Problem is linked to aperiodic tilings. This implies a connection between the global symmetries of the tiling and the computational complexity of the problem. The preceding theorem shows that the Wang Tile problem goes from undecidable to trivial if the tiles can be rotated. So, there is also a connection between the local symmetry of a single tile and the computational complexity of the problem.

The remainder of this chapter looks at different techniques for modelling inifinite tiling problems as graphs. Particular attention is given to restricting the group of automorphisms of the graph of a single tile and the relationship between this group and the group of symmetries of a corresponding geometric tile.

4.2 Building an Infinite Graph from a Single Tile

In order to extend our graph theoretical model to the study of infinite tiling problems we must start with an infinite graph. When trying to model a problem from geometry, a good way to build the graph is by taking the planar dual of the faces of a tiling of uncolored dominos. Figure 4.2 illustrates this process for the Wang Tile. Recall that this is the method we used to get the graph underlying domino tilings in Chapter 3. The case of dominos shows that this method can work even

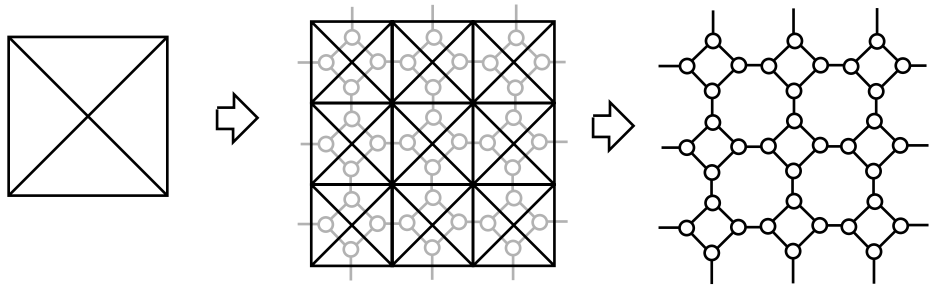

Figure 4.2: Constructing an infinite graph from the planar dual of a tiling.

when the tiles can tiles the plane in several different ways as long as all such tilings have the same dual graph.

It is important to note that for many cases, the group of symmetries of the tile corresponds exactly to the group of automorphisms of that tile's graph representation.

4.2.1 Restricting Tile Symmetries

One way to restrict the symmetries of a graph is to add vertices and edges.

Theorem 4.2.1. *The addition of three vertices suffices to trivialize the group of symmetries for a simply connected tile.*

Proof. The faces of a simply connected tile correspond to segments of the simple closed curve that bounds the tile. So, the tile can be drawn so that $G(V, E)$, the planar dual of the tile is a cycle on n vertices where n is the number of tile faces. We pick two vertices u, v in the graph representing the tile. We add a leaf vertex adjacent to u and a path of length 2 adjacent to v. The group of automorphisms of the original cycle is just the dihedral group D_n. The modified graph. Let $\Phi : V \to V$ be an automorphism of the tile graph. The vertices u, v are the only vertices of degree 3. It cannot be the case that $\Phi(u) = v$ because u is adjacent to a vertex of degree 1 and v is not. So, it must be that both u and v are fixed by Φ. If we restrict Φ to the original cycle, we get an automorphism with two fixed points, u and v. The only element of D_n with two fixed points is the identitiy. The map Φ must also fix the new vertices added so we see that Φ is the identity automorphism.

□

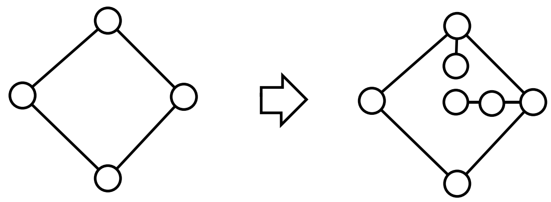

Figure 4.3: Trivializing the group of symmetries of tile graph.

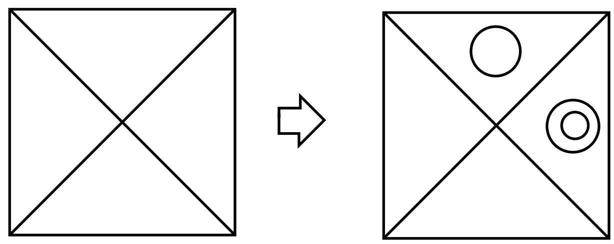

Figure 4.4: Geometric interpretation of tile augmentation.

The addition of vertices to the graph has a simple geometric interpretation as shown in Figure 4.4. We can imagine that the top face is marked with a circle and the right face is marked with a double circle. The augmented graph is exactly the planar dual of the resulting drawing.

Chapter 5

Conclusions

We have shown that the Domino Tiling Problem as stated by Watson and Worman [14] is NP-Complete. We have also presented a graph theoretical model that describes certain geometric tiing problems and used the model to prove other interesting results about tilings. The graph theoretical Domino Tiling Problem was explored in depth and a variety of subproblems were shown to be NP-Complete as well. In the wake of this wave of negative result, perhaps the most important open question that remains is whether or not there exist nontrivial variants of the domino tiling problem that yield polynomial-time solutions.

We also expanded the graph tiling model to describe infinite tiling problems. In particular, we looked at the ways that the automorphisms of a tile graph can correspond to the symmetries of the geometric tile that it models. Motivated by the connection between the local symmetries of a tile, the global symmetries of a tiling, and the tractability of the tiling, a method for restricting the symmetries of a tile was presented. The majority or research done on undecidable domino tiling problems are based on the Wang Tile. Hopefully, this research will lay the groundwork for more general graph theoretical methods that can show other that tiling the plane with other types of tiles is also undecidable.

All of the geometric tilings addressed in this paper are known as *monohedral* tilings because the tiles are all congruent. However, the addition of colors on the faces of the tiles allows there to be more then one type of tile. Consequently, there exist aperiodic sets of colored monehedral tiles, a phenomenon that is not known to exist for uncolored monohedral tilings.

Section 3.5 we give an alternative to defining valid adjacency by coloring the faces of the tiles. This opens up a variety of new and interesting questions. Specifically, it would be worthwhile to study the affects on the symmetries of infinite tilings given the new notion of coloring. We conjecture that there exists undecidable sets of rotatable Wang Tiles subject to some adjacency graph G_A.

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