Cone Depth and the Center Vertex Theorem

- Gary Miller
- Todd Phillips
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Let $P$ be $n$ points in $\mathbb{R}^d$.

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• Simplicial depth
• Convex hull peeling
• Regression Depth
• k-order α-hulls
• Travel Depth
• ...... many others
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Can we pick a center from \( P \)?
Can we pick a center from P?

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\[ CD(x) = \min_{||v||=1} |\{ p \in P \mid \frac{(p-x)v}{||p-x||} > c \}| \]
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For this talk: cones have half-angle 45°
a center vertex is a point \( p \in P \) such that \( CD(p) \geq n/d+1 \).

**Thm:** For all \( P \subset \mathbb{R}^d \), there exists a center vertex.

**Pf:** Pick \( p \in P \) closest to a center point.
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In $\mathbb{R}^d$, the idea is the same.

Pick the right hyperplane through the center point, $c$.

Show that the bounded part of the cone is empty.

The "right" hyperplane is the one that intersects the cone at a hyperellipsoid centered at $c$. 
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Beyond the plane

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Pick the right hyperplane through the center point, $c$.

Show that the bounded part of the cone is empty.

The “right” hyperplane is the one that intersects the cone at a hyperellipsoid centered at $c$. 
Let $p_k$ be the $k$-th nearest point to $c$.

$$CD(p_k) \geq (n/d+1) - (k-1)$$

So, $p_1, \ldots, p_{n/2(d+1)}$ have depth at least $n/2(d+1)$.

Thus, the average depth is at least 
$$n/4(d+1)^2 = O(n).$$
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Let $p_k$ be the $k$-th nearest point to $c$.

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So, $p_1, \ldots, p_{n/2(d+1)}$ have depth at least $n/2(d+1)$.

Thus, the average depth is at least $\frac{n}{4(d+1)^2} = O(n)$. 
Some open questions.

- Is $45^\circ$ the largest cone half-angle for which a center vertex *always* exists?
- How fast can we compute the cone depth of a point in space?
- How fast can we find a center vertex deterministically.
Thanks.