Learning with Nets and Meshes

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Thanks to my collaborators:

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Gary Miller
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Steve Oudot
Point Clouds in low to medium dimensional ambient space.

Rule of thumb: $d!$ or $2^{d^2}$ is okay but $n^d$ is not.
There are many geometric inference problems that could benefit from meshing.

Discretize Space
Approximate Functions
Adapt to density
Describe the space *around* the input.
Meshing

Input: \( P \subset \mathbb{R}^d \) \( n = |P| \)
Output: \( M \supset P \) with a “nice” Voronoi diagram
Meshing

Input: $P \subset \mathbb{R}^d$, $n = |P|$
Output: $M \supset P$ with a “nice” Voronoi diagram
Points, offsets, homology, and persistence.
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offsets

Compute the Homology
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Offsets

Persistent

Compute the Homology
Geometric Approximation

Topologically Equivalent
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Complexity: How big is the mesh?
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How many Steiner Points?
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How many simplices?
Complexity: How big is the mesh?

How many Steiner Points?

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How many simplices?

Only \( O(|M|) \) simplices.

Compare to \( |M|^{\lceil d/2 \rceil} \) for general Delaunay triangulations.

Constants depend on aspect ratio.
Complexity: How hard is it to compute a mesh?
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Point Location
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Point Location  
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$O(n \log n)$

Compute a *hierarchical quality* mesh.
Complexity: How hard is it to compute a mesh?

$O(n \log n + |M|)$

This is optimal in the comparison model.

$O(n \log n) + O(|M|)$

Compute a *hierarchical quality* mesh. Finishing post-process.
The Delaunay Triangulation is the dual of the Voronoi Diagram.
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No D-ball intersects more than \(O(1)\) others.
D-Balls are Constant Ply
Total Complexity is linear in $|M|$ 
No D-ball intersects more than $O(1)$ others.
Refine poor-quality cells by adding a Steiner point at its farthest vertex.
We replace *quality* with *hierarchical quality*.
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Inside the cage: Old definition of quality.
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Inside the cage: Old definition of quality. Outside: Treat the whole cage as a single object.
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Inside the cage: Old definition of quality. 
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All the same properties as quality meshes: ply, degree, etc.
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1. Build a Voronoi diagram incrementally.
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3. Recover quality after each input point.
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This is how we avoid the worst-case Voronoi bounds.
Point Location and Geometric D&C

Idea: Store the uninserted points in the D-balls. When the balls change, make local updates.
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Lemma (HMP06). A point is touched at most a constant number of times before the radius of the largest D-ball containing it goes down by a factor of 2.
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Geometric Divide and Conquer: $O(n \log \Delta)$
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Old: Progress was decreasing the radius by a factor of 2.
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Goal: Combinatorial Divide and Conquer

Old: Progress was decreasing the radius by a factor of 2.

New: Decrease number of points in a ball by a factor of 2.
“Nets catch everything that’s big.”
Range Nets

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Definition. A range space is a pair \((X, R)\), where \(X\) is a set (the vertices) and \(R\) is a collection of subsets (the ranges).
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Definition. Given a range space \((X, R)\), a set \(N \subseteq X\) is a range space \(\varepsilon\)-net if for all ranges \(r \in R\) that contain at least \(\varepsilon|X|\) vertices, \(r\) contains a vertex from \(N\).
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**Theorem:** [Chazelle & Matousek 96] For \(\varepsilon, d\) fixed constants, \(\varepsilon\)-nets of size \(O(1)\) can be computed in \(O(n)\) deterministic time.
For each D-Ball, select a $1/2d$-net of the points it contains.
Take the union of these nets and call it a round.
Insert these.
Repeat.
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**Lemma.** Let $M$ be a set of vertices. If an open ball $B$ contains no points of $M$, then $B$ is contained in the union of $d$ D-balls of $M$. 
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$\log(n)$ Rounds
Amortized Cost of a Round is $O(n)$

Watch an uninserted point $x$.
Claim: $x$ only gets touched $O(1)$ times per round.
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$y$ touches $x$ $\Rightarrow$ $y$ is “close” to $x$

close $= 2$ hops among $D$-balls
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$O(n)$ total work per round.
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There are many possible new applications, waiting to be discovered (i.e. any time you would have used a Voronoi diagram).
Thanks.