Persistent Homology and Nested Dissection

Don Sheehy
University of Connecticut

joint work with Michael Kerber and Primoz Skraba

A Topological Data Analysis Pipeline

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Data Points

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Data Points → a Function (Lipschitz)

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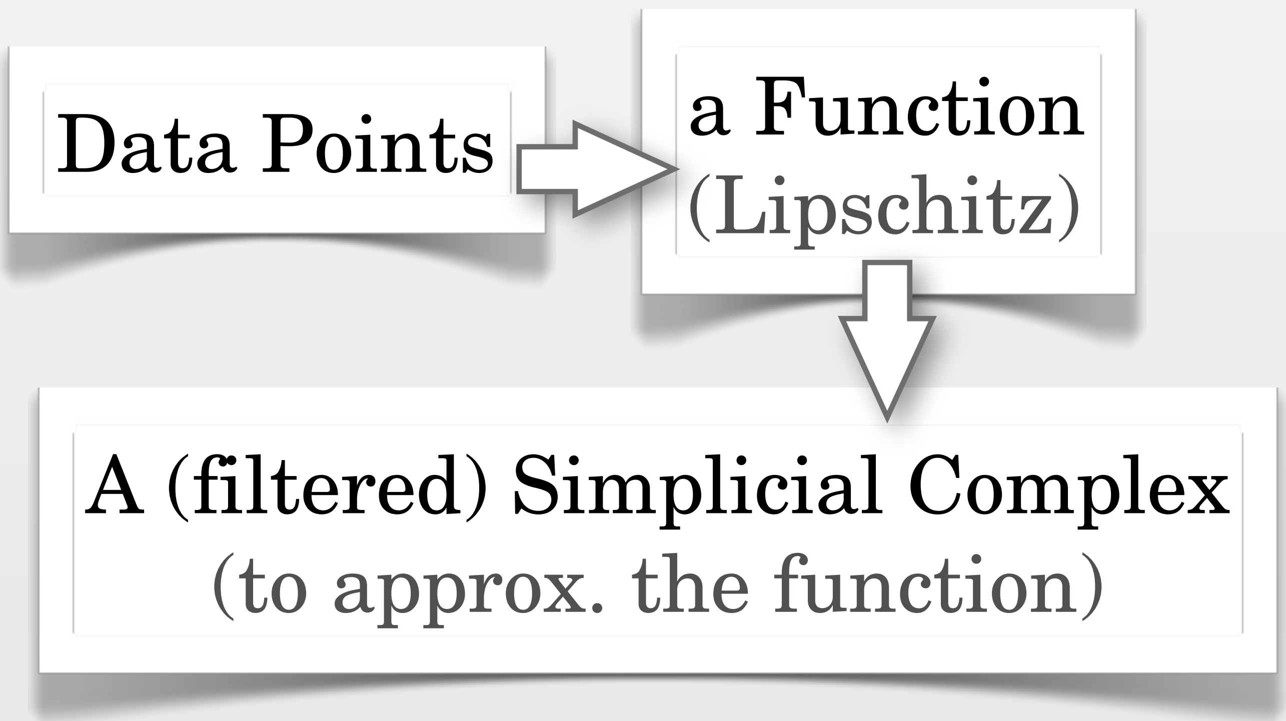

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A (filtered) Simplicial Complex
(to approx. the function)

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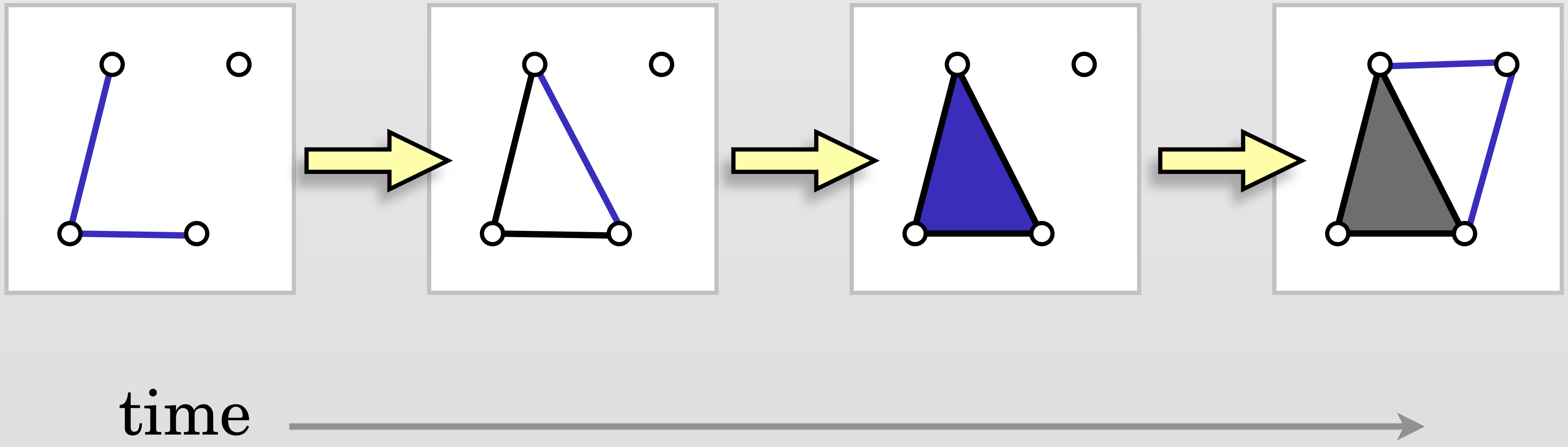
How to compute it?

Q: Can we use the geometry to speed up the persistence computation?

We want to build a filtered simplicial complex.

Associate a birth time with each simplex in complex K.

At time α, we have a complex K_α consisting of all simplices born at or before time α.



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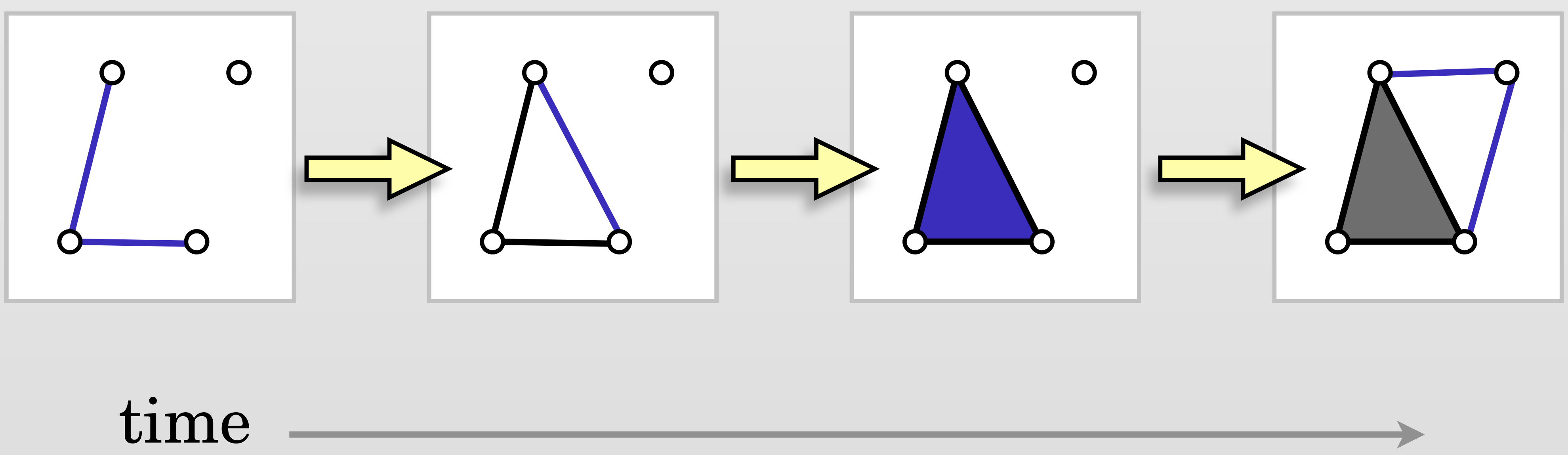

This can be represented by its boundary matrix D, with a fixed row order.

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It's just Gaussian elimination!

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Clear and Compress [BKR13]

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The Punchline:
Inverting A can be done in $O(n^{\beta\omega})$ time.
Also works for computing ranks of singular, nonsymmetric matrices over finite fields.

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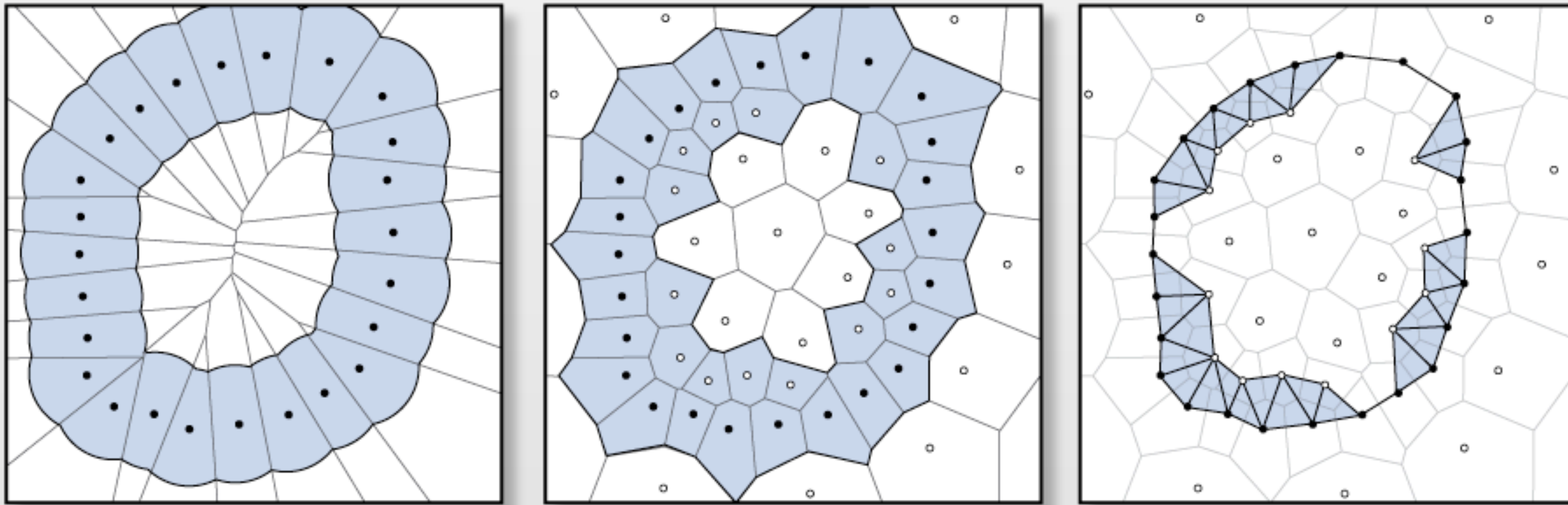
The first trick:
In the output-sensitive persistence algorithm of Chen and Kerber, only ranks need to be computed.

The other trick:
The geometric separators of Miller-Teng-Thurston-Vavasis can also give separators for minors of the boundary matrix of a quality mesh.

Mesh Filtrations

Geometric Approximation | Topologically Equivalent



1. Compute the function on the vertices.

2. Approximate the sublevel with a union of Voronoi cells.

3. Filter the Delaunay triangulation appropriately.

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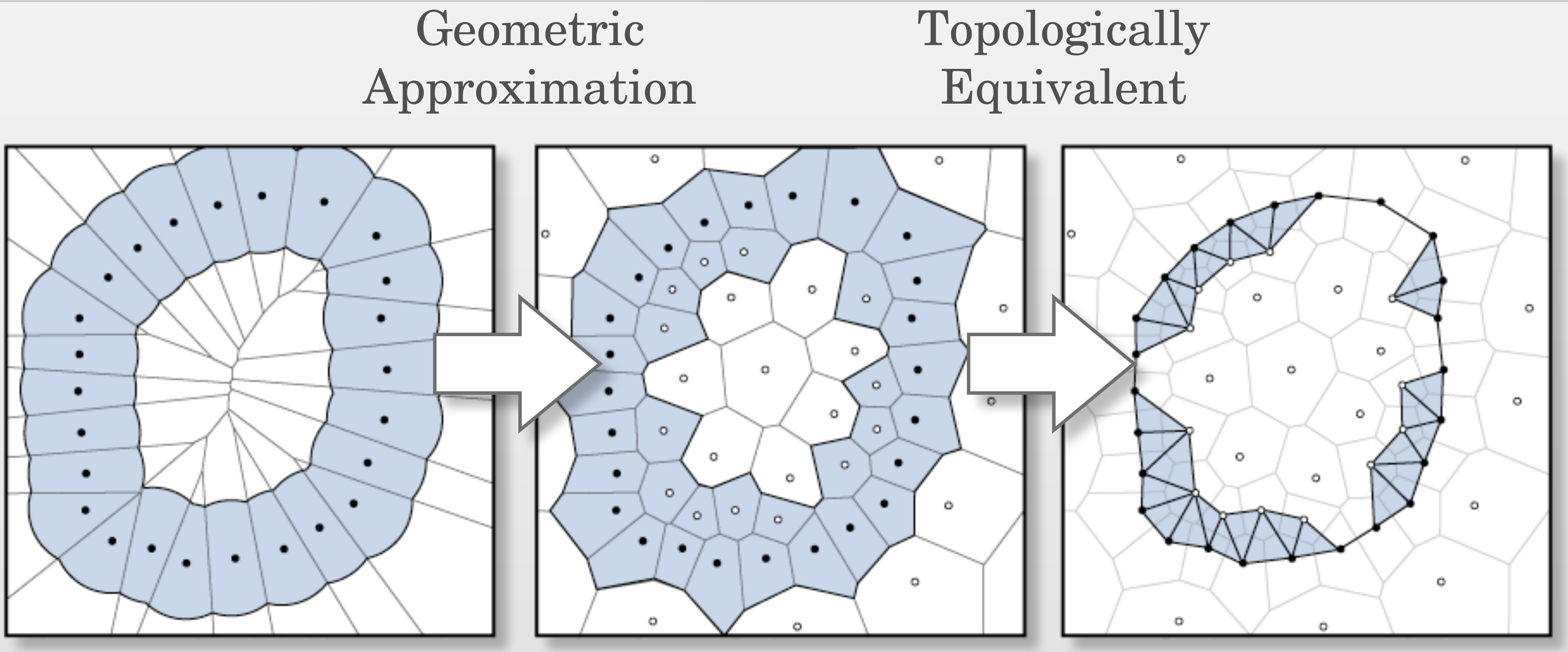

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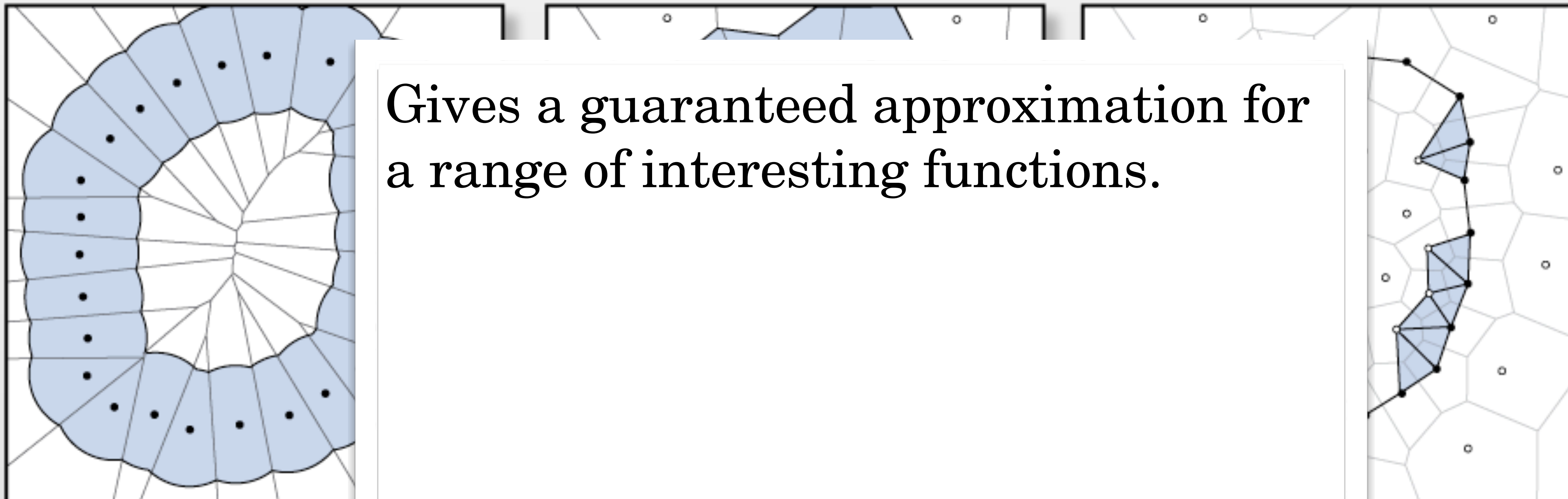

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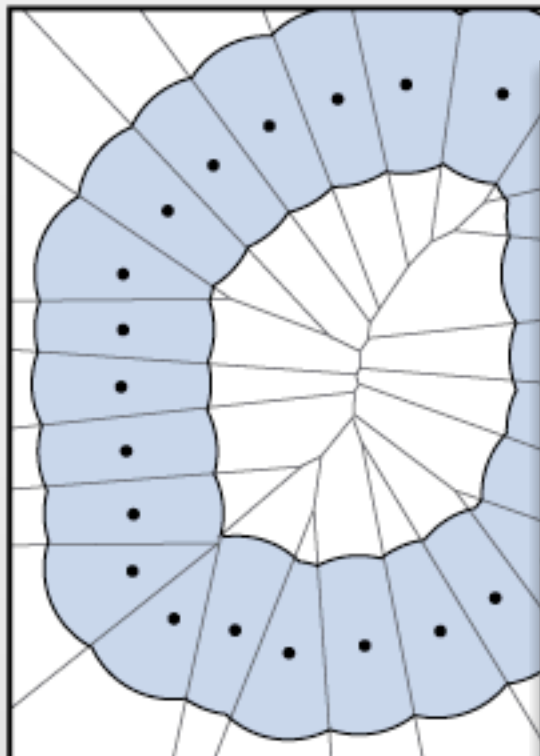

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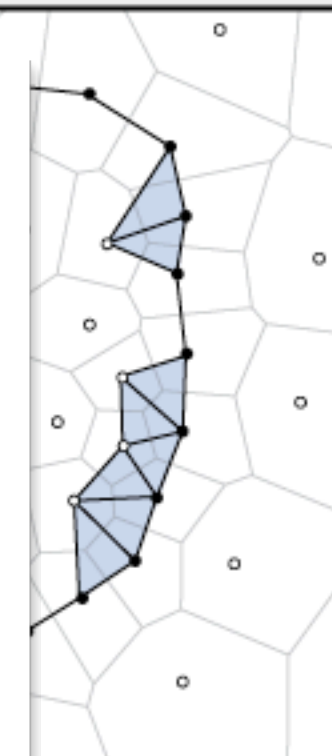
Bonus: The theory of geometric separators was invented for graphs coming from meshes like this.



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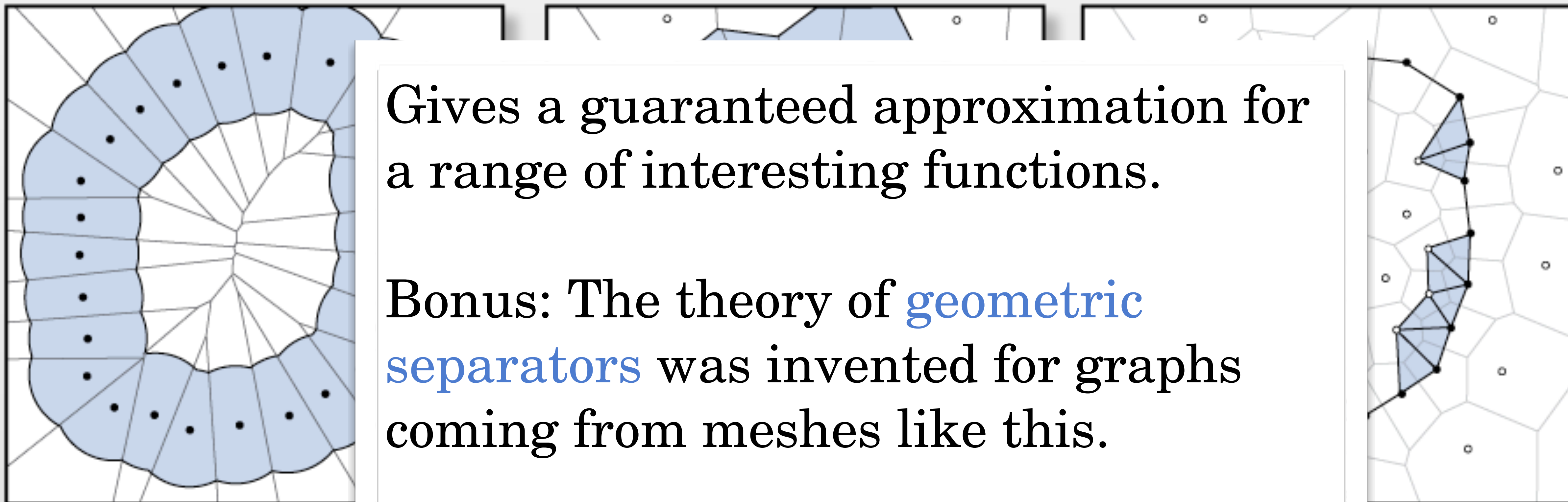

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We extend the geometric separators to complex separators.

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Geometric Separators

Nested Dissection

Output-Sensitive Persistence Algorithm

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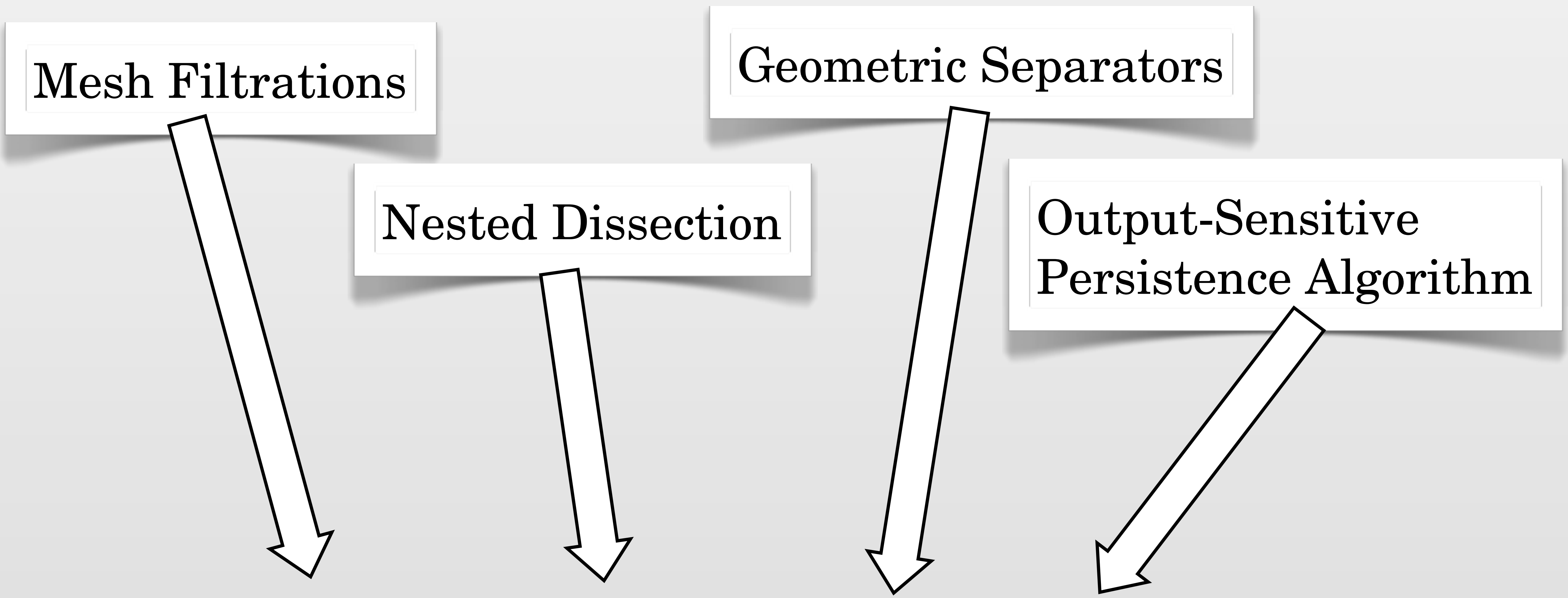

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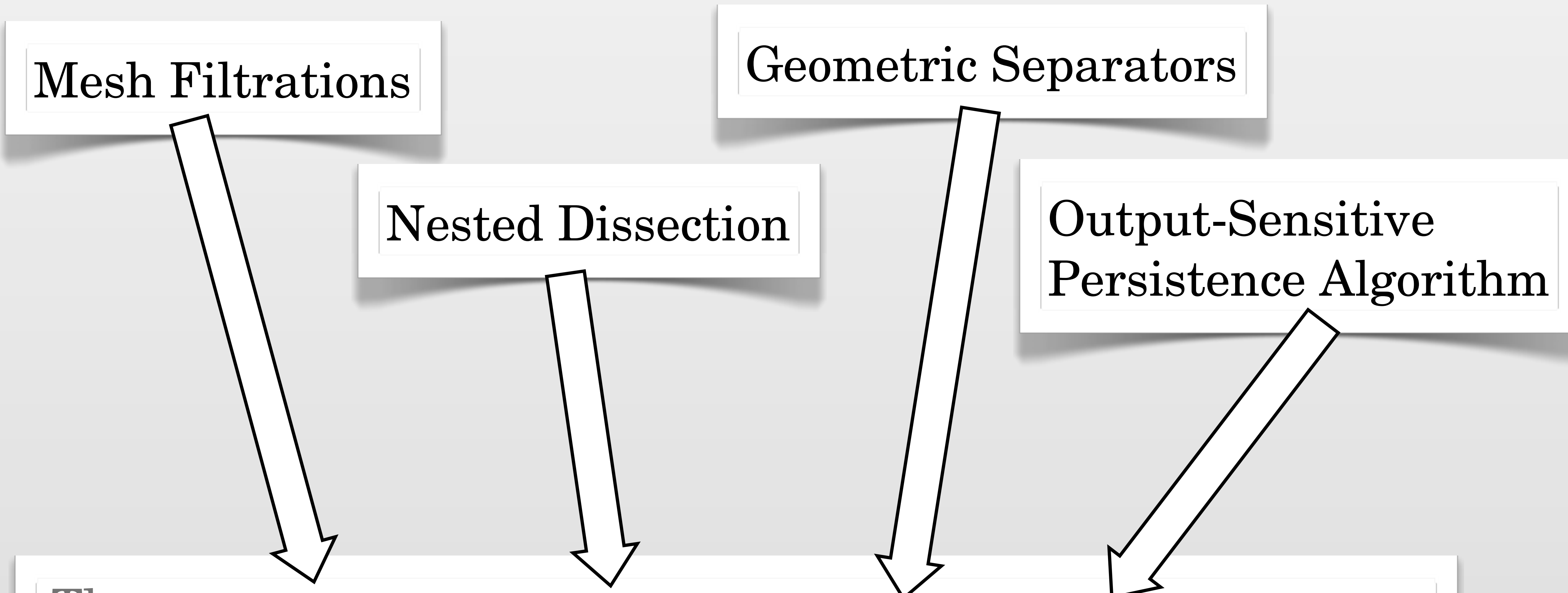

Theorem.

Let F be a filtration on the Delaunay triangulation of a set of τ-well-spaced points. For a constant $\Gamma \geq 0$, the subset D_Γ of the persistence diagram of F consisting of those pairs with persistence at least Γ can be computed in $O(|D_{(1-\delta)\Gamma}|n^{\omega(1-\frac{1}{d})})$ time, where $D_{(1-\delta)\Gamma}$ is the set of pairs in the persistence diagram with persistence at least $(1-\delta)\Gamma$ for any constant $\delta > 0$.

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Thanks.