Planar Graphs in $2^{1/2}$ Dimensions

Don Sheehy
$2^{1/2}$ Dimensions
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Cast of Characters
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James Clerk Maxwell

Luigi Cremona
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Ernst Steinitz
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James Clerk Maxwell
Luigi Cremona
Ernst Steinitz
W. T. Tutte
Planar Graphs
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Planar Graphs
Duality
Duality

F

E

V
Duality

F

E

V

V

E

F
Polar Polytopes

\[ A^\circ = \{ x \in \mathbb{R}^d \mid a \cdot x \leq 1, \forall a \in A \} \]
The Maxwell-Cremona Correspondence
Equilibrium Stresses
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There is a 1-1 correspondence between “proper” liftings and equilibrium stresses of a planar straight line graph.
The Maxwell-Cremona Correspondence
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Reciprocal Diagrams from Equilibrium Stresses
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Reciprocal Diagrams from Liftings
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The Maxwell-Cremona Correspondence

Equilibrium Stresses

Reciprocal Diagrams

Liftings
Other Famous Reciprocal Diagrams

Delaunay Triangulation

Voronoi Diagram
Other Famous Reciprocal Diagrams

Weighted Delaunay Triangulation

Weighted Voronoi Diagram
Other Famous Reciprocal Diagrams

**Weighted** Delaunay Triangulation

2½ dimensional polarity

**Weighted** Voronoi Diagram
How to Draw a Graph
Tutte’s Algorithm

1. Fix one face of a simple, planar, 3-connected graph in convex position.
2. Place each other vertex at the barycenter (centroid) of its neighbors.

The result is a non-crossing, convex drawing.
Spring Interpretation
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Computing Forces

$v \in \mathbb{R}^2$
Computing Forces

\[ \mathbf{F}_v = \sum_{u \sim v} (v - u) \]

\( v \in \mathbb{R}^2 \)
Computing Forces

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\[ = d_v v - \sum_{u \sim v} u \]
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Computing Forces

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$= d_v v - \sum_{u \sim v} u$

$F = LV$

$L = D - A$
Computing Forces

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degrees
Computing Forces

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\[ = d_v v - \sum_{u \sim v} u \]

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Computing Forces

\[ v \in \mathbb{R}^2 \]

\[ F_v = \sum_{u \sim v} (v - u) = d_v v - \sum_{u \sim v} u \]

\[ F = LV \]

\[ L = D - A \]

The Laplacian!
Computing Forces

\[ LV = F \]
Computing Forces

\[ LV = F = 0? \]
Computing Forces

\[ L V = F = 0? \]

\( V_1: \) boundary
\( V_2: \) interior
Computing Forces

\[ LV = F = 0? \]

\[ V_1: \text{boundary} \]
\[ V_2: \text{interior} \]

\[
\begin{bmatrix}
L_1 & B^T \\
B & L_2
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= 
\begin{bmatrix}
F' \\
0
\end{bmatrix}
\]
Computing Forces

$$LV = F = 0? \quad V_1: \text{boundary}$$
$$V_2: \text{interior}$$

$$\begin{bmatrix} L_1 & B^T \\ B & L_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} F' \\ 0 \end{bmatrix}$$

$$BV_1 + L_2V_2 = 0$$
**Computing Forces**

\[ LV = F = 0? \]

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Monotone Paths

Pick a direction and a vertex. There is a monotone path in that direction from the vertex to the boundary.
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Planar, 3-Connected Graphs
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→ No $K_5$ or $K_{3,3}$ minors
Planar, 3-Connected Graphs

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- Removing a face does not disconnect the graph.
Planar, 3-Connected Graphs

\[ \Rightarrow \text{No } K_5 \text{ or } K_{3,3} \text{ minors} \]

\[ \Rightarrow \text{Removing a face does not disconnect the graph.} \]

\[ \Rightarrow \text{No face has a diagonal.} \]
Lemma: No two disjoint paths have interleaved endpoints on a face.
Double Crossing a Face

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Tutte’s Algorithm

No ZigZags
Tutte’s Algorithm

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No ZigZags
Tutte’s Algorithm

No ZigZags
Tutte’s Algorithm

No Crossings
Tutte’s Algorithm

No Crossings
Tutte’s Algorithm

No Crossings
Tutte’s Algorithm

No Crossings
Tutte’s Algorithm

No Crossings
Tutte’s Algorithm

No Crossings
Tutte’s Algorithm

No Crossings
Tutte’s Algorithm

No Overlaps
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No Overlaps
Weirdness on the outer face.
Weirdness on the outer face.

Lifting still works, except outer face.
Weirdness on the outer face.
Lifting still works, except outer face.
Lifting is convex.
Steinitz’s Theorem
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A graph $G$ is the 1-skeleton of a 3-polytope if and only if it is simple, planar, and 3-connected.
Claim: If the graph has a triangle, then the Tutte embedding followed by the Maxwell-Cremona lifting gives the desired polytope.
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Fix the triangle as the outer face.
Steinitz’s Theorem

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Fix the triangle as the outer face.

After the lifting, the triangle must lie on a plane.
Question: What if there is no triangle?
Steinitz’s Theorem

**Question:** What if there is no triangle?

**Answer:** Dualize (the dual has a triangle)
Steinitz’s Theorem

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.
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(No triangles)

\[ \frac{|E|}{2} - |E| + \frac{|E|}{2} \geq 2 \]

\[ 0 \geq 2 \]
So, with the Tutte embedding and the Maxwell-Cremona Correspondence, we can construct a polytope with 1-skeleton isomorphic to either the graph or its dual.
Steinitz’s Theorem

So, with the **Tutte embedding** and the **Maxwell-Cremona Correspondence**, we can construct a polytope with 1-skeleton isomorphic to *either* the graph *or* its dual.

If we have the dual, *polarize*.
So, with the **Tutte embedding** and the **Maxwell-Cremona Correspondence**, we can construct a polytope with 1-skeleton isomorphic to *either* the graph *or* its dual.

If we have the dual, *polarize*.  

[Steinitz's Theorem]

[Eades, Garvan 1995]
A Tour of Other Stuff
Rigidity and Unfolding

[Connelly, Demaine, Rote, 2000]
Greedy Routing

[Morin, 2001]

[Papadimitriou, Ratajczak, 2004]
Robust Geometric Computing

[Hopcroft and Kahn 1992]
Correspondence between Colin de Verdiere matrices and Steinitz representations

[Lovasz, 2000]
Correspondence between Colin de Verdiere matrices and Steinitz representations

[Lovasz, 2000]

Lemma 4 We can assign a vector $w_f$ to each $f \in V^*$ so that whenever $ij \in E$ and $fg$ is corresponding edge of $G^*$, then

$$w_f - w_g = M_{ij}(u_i \times u_j).$$

(2)

Proof. Let $v_{fg} = M_{ij}(u_i \times u_j)$. It suffices to show that the vectors $v_{fg}$ sum to 0 over the edges of any cycle in $G^*$. Since $G^*$ is a planar graph, it suffices to verify this for the facets of $G^*$. Expressing this in terms of the edges of $G$, it suffices to show that

$$\sum_{j \in N(i)} M_{ij}(u_i \times u_j) = 0$$

(where, as usual, $N(i)$ denotes the set of neighbors of $i$). But this follows from (1) upon multiplying by $u_i$, taking into account that $u_i \times u_i = 0$ and $M_{ij} = 0$ for $j \notin N(i) \cup \{i\}$. □
Thank you.
Thank you.

Questions?