Beating the Spread: Time-Optimal Point Meshing

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Carnegie Mellon
(soon: INRIA)

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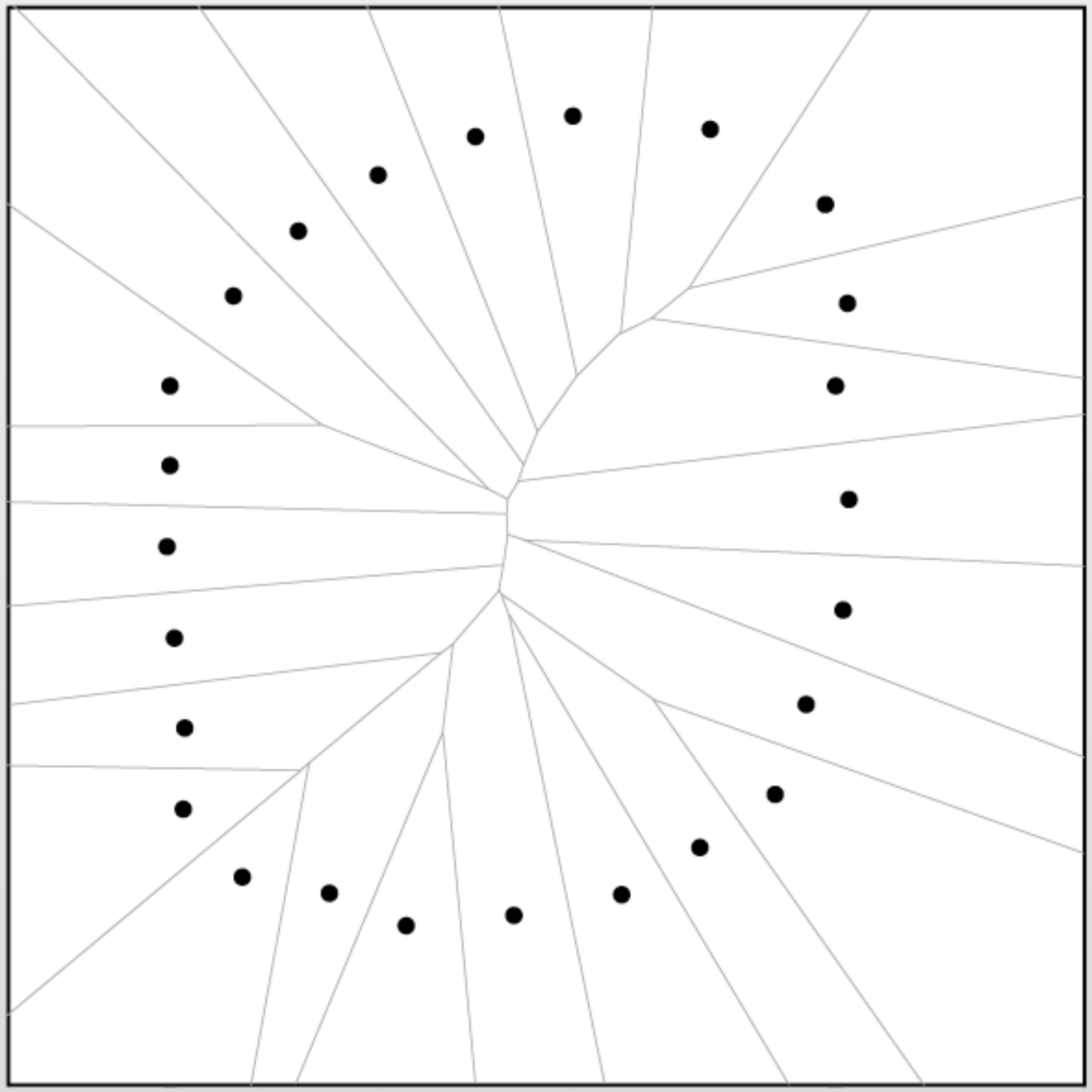
Meshing Points

Input: $P \subset \mathbb{R}^d$

Output: $M \supset P$ with a “nice” Voronoi diagram

$n = |P|, m = |M|$



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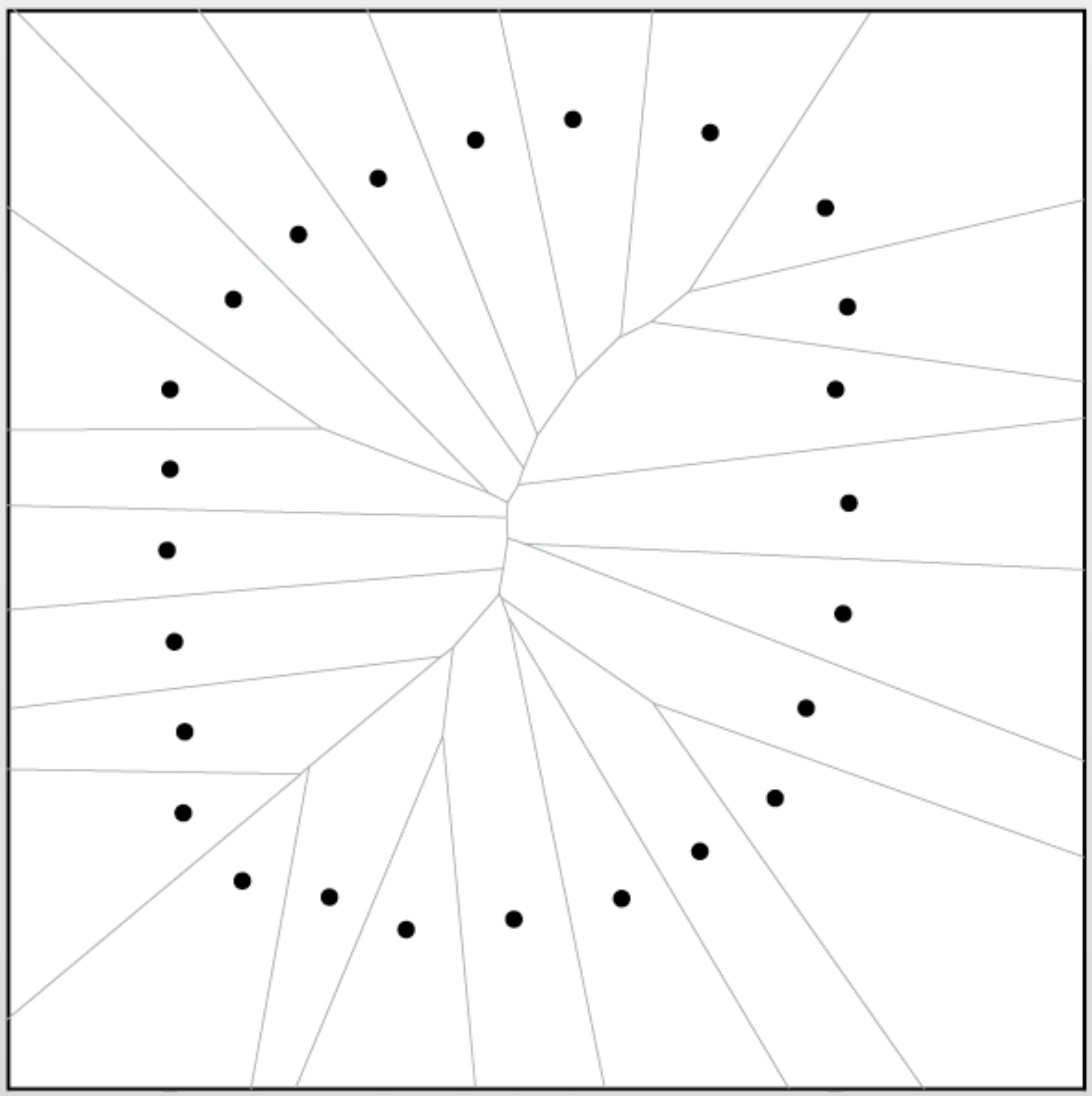
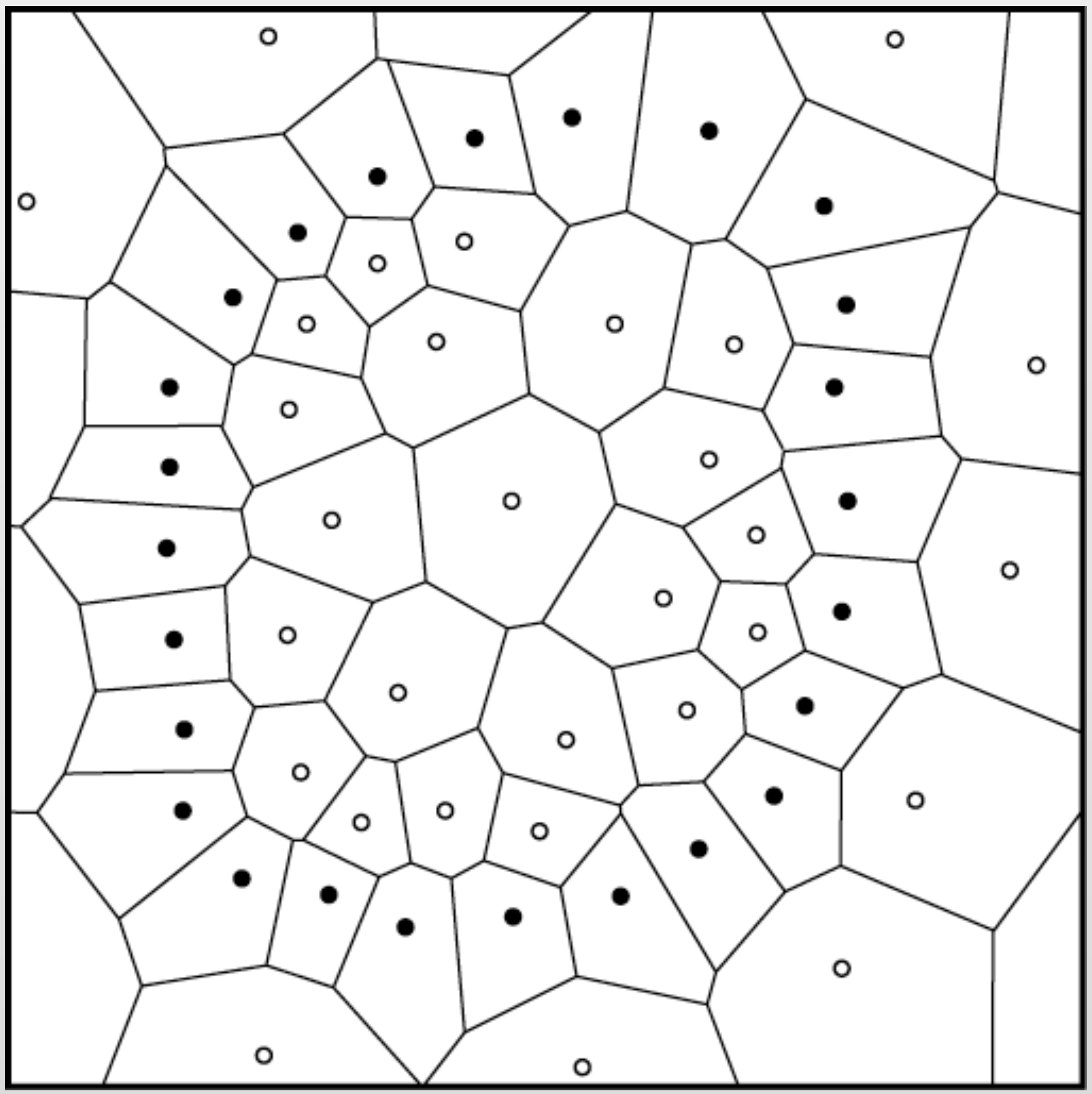
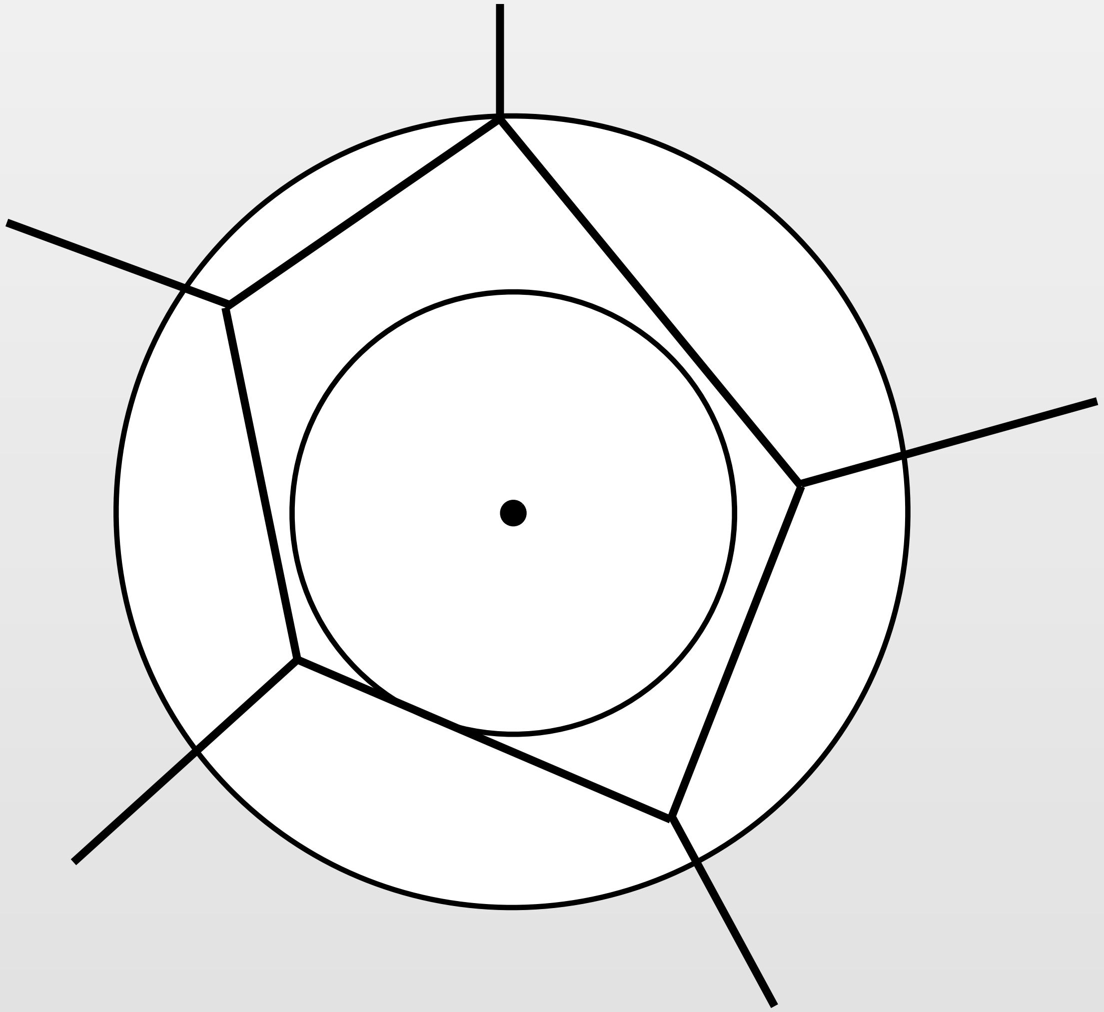




Quality Meshes have cells with bounded *aspect ratio*.

$$\text{aspect ratio} = \frac{\mathrm{R}}{\mathrm{r}} \leq \tau$$



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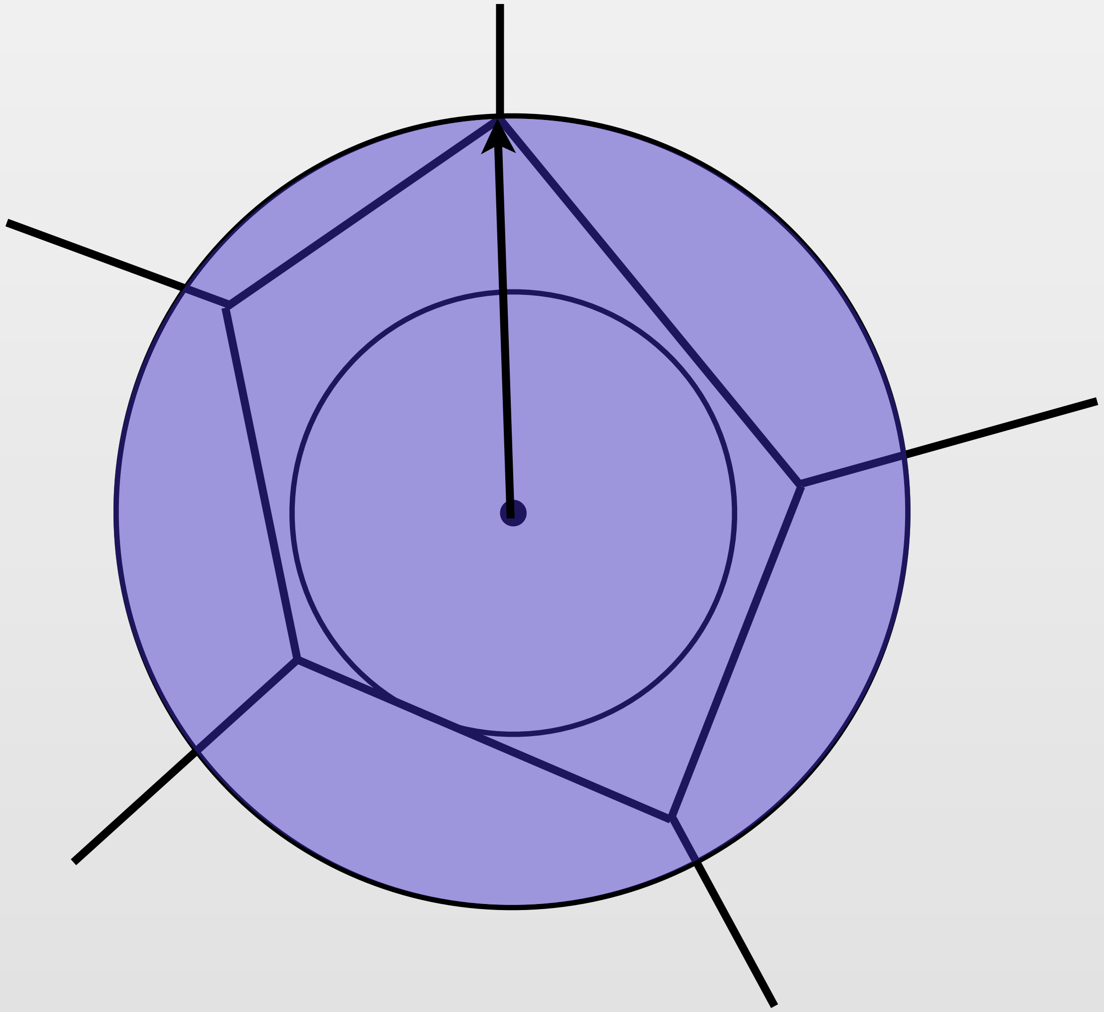

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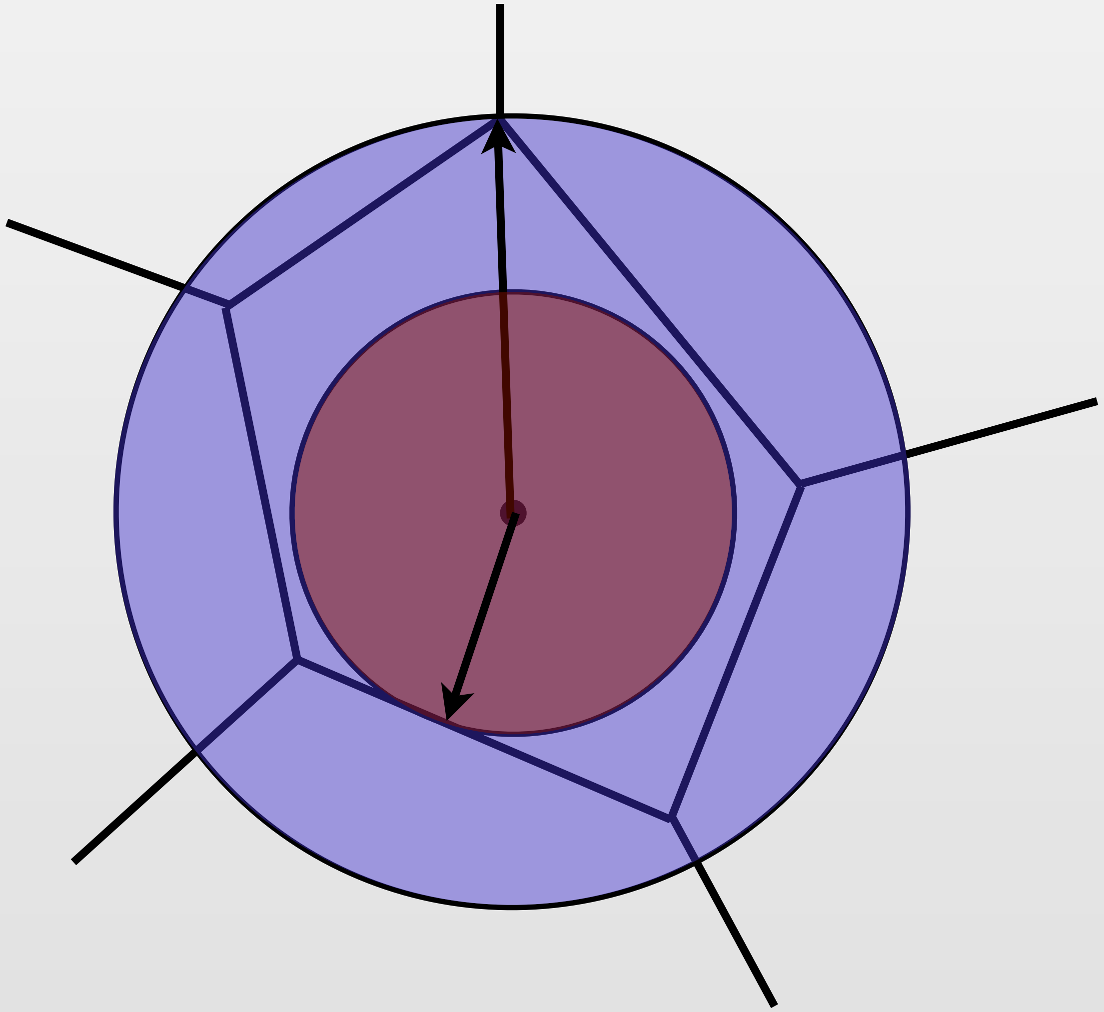

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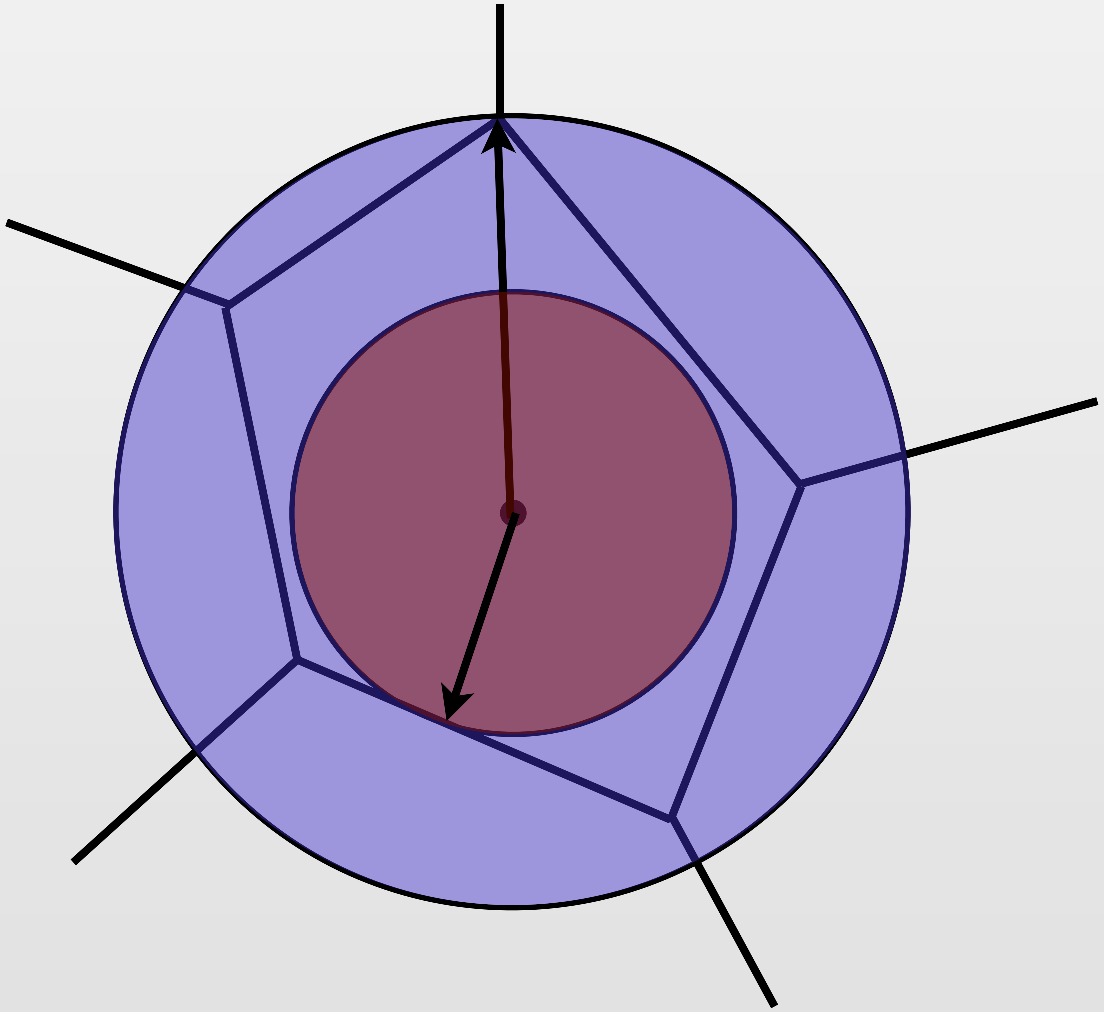

$$\boxed{\tau \geq 2 + \varepsilon}$$

Prior Work

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Delaunay Refinement:

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Chew '89 .... 2D

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Hides dimension terms

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For some point sets, $m = \Omega(n \log \Delta)$

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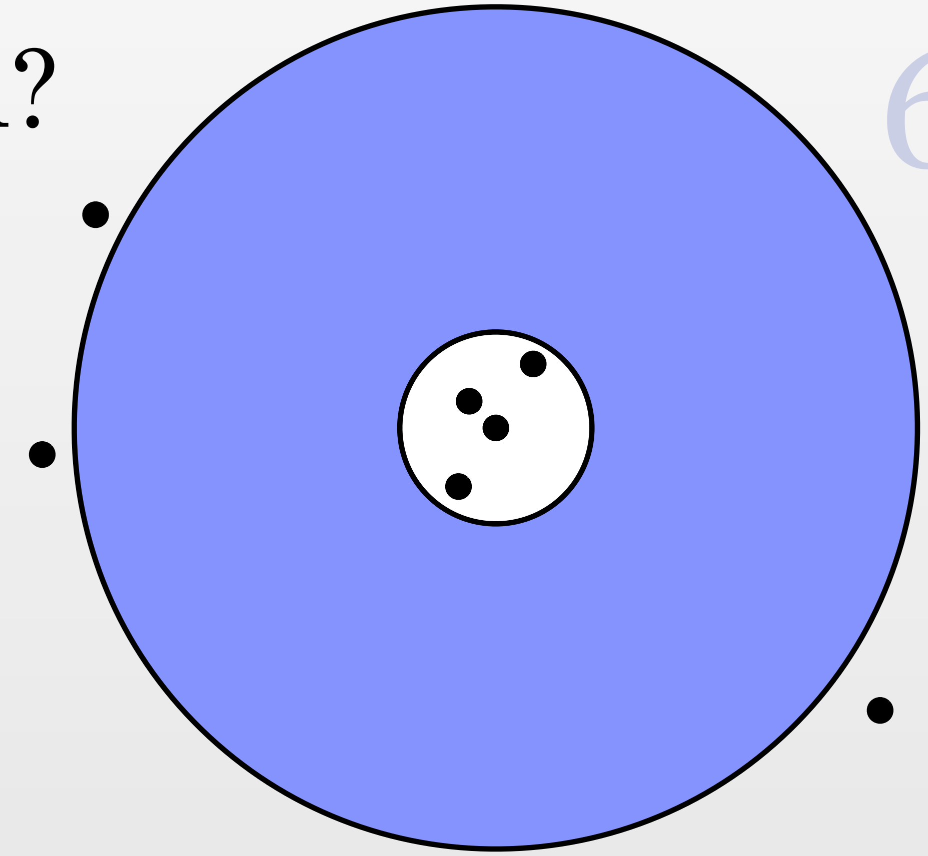

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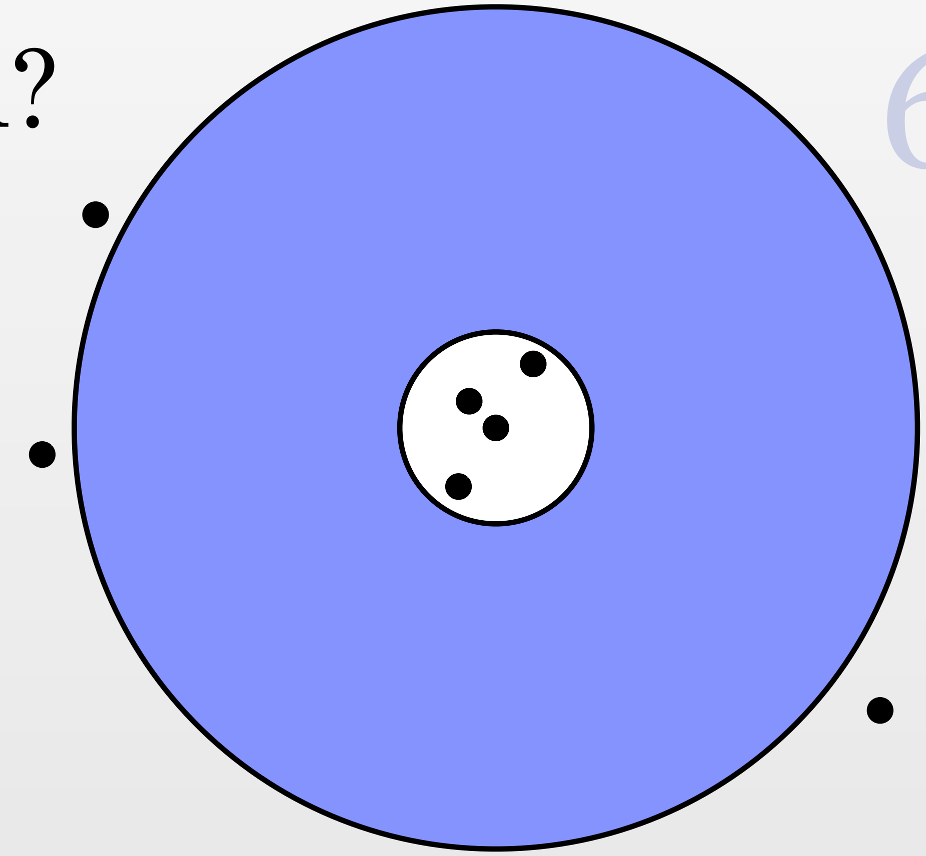

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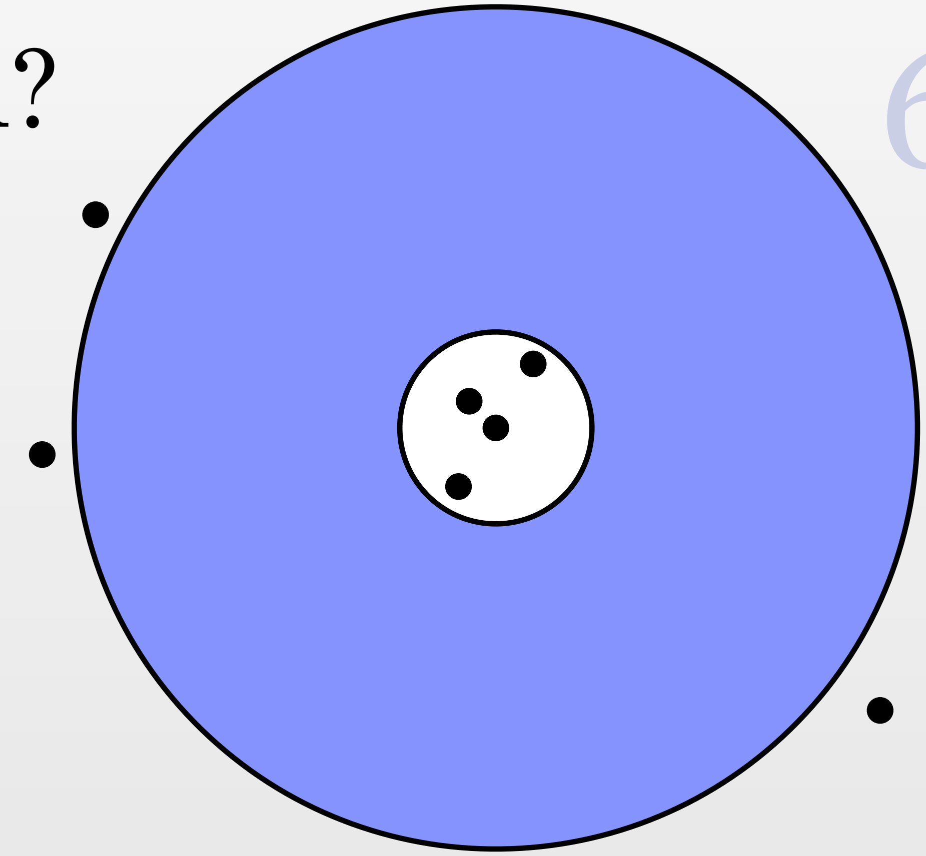

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How many simplices?

Only $O(m)$ simplices.
Compare to $m^{\lceil d/2 \rceil}$ for general Delaunay triangulations.
Constants depend on aspect ratio.

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Finishing post-process.
(easy)

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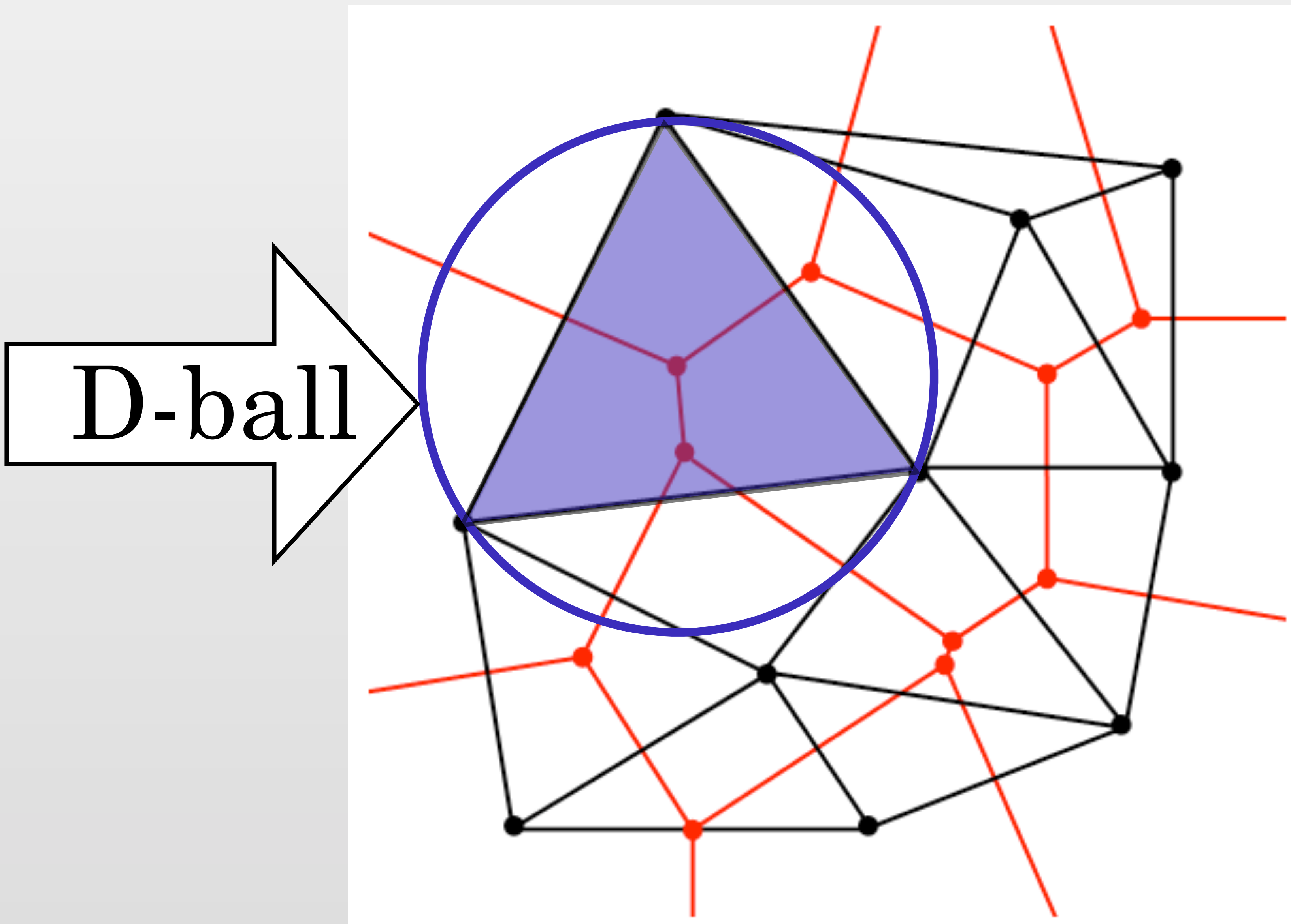
The Delaunay Triangulation is the dual of the Voronoi Diagram.



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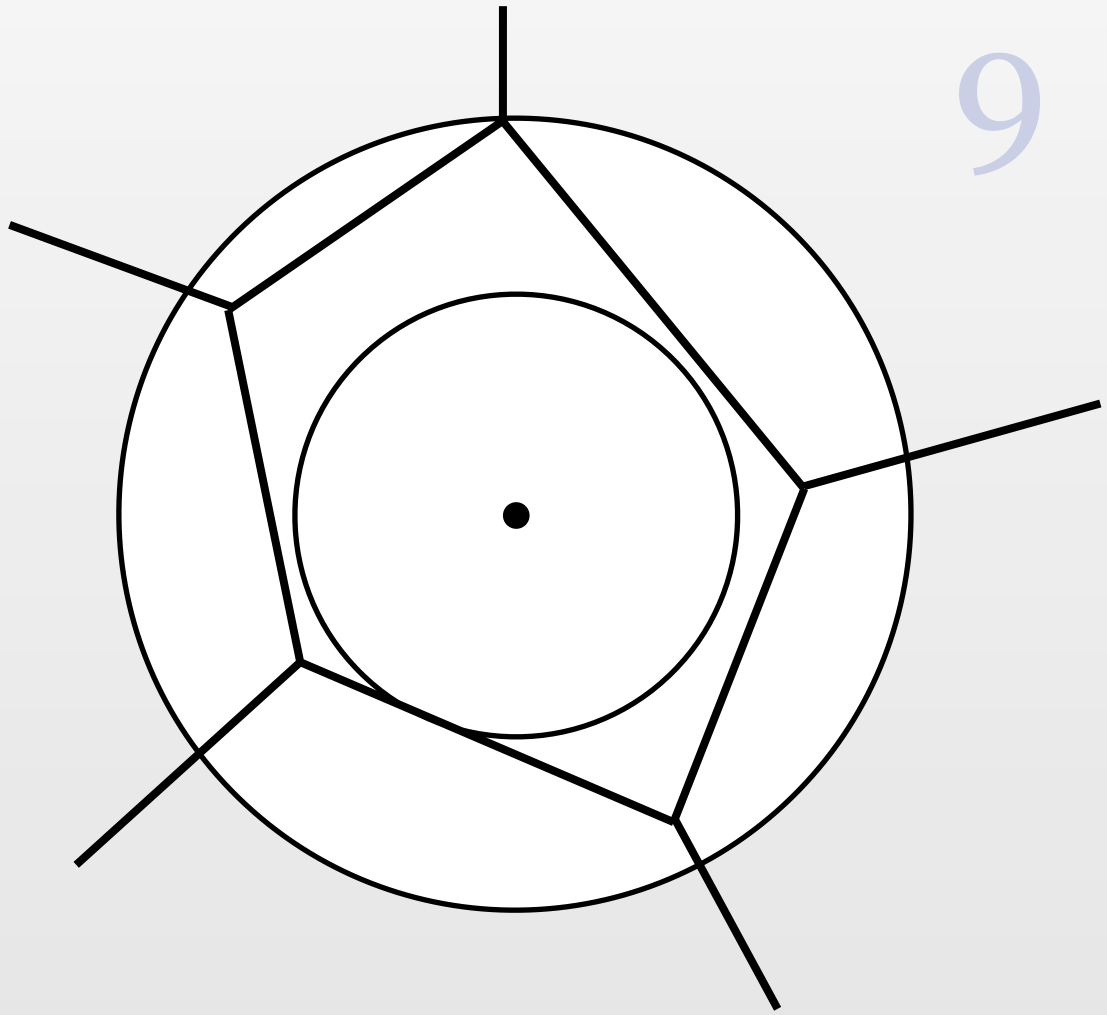

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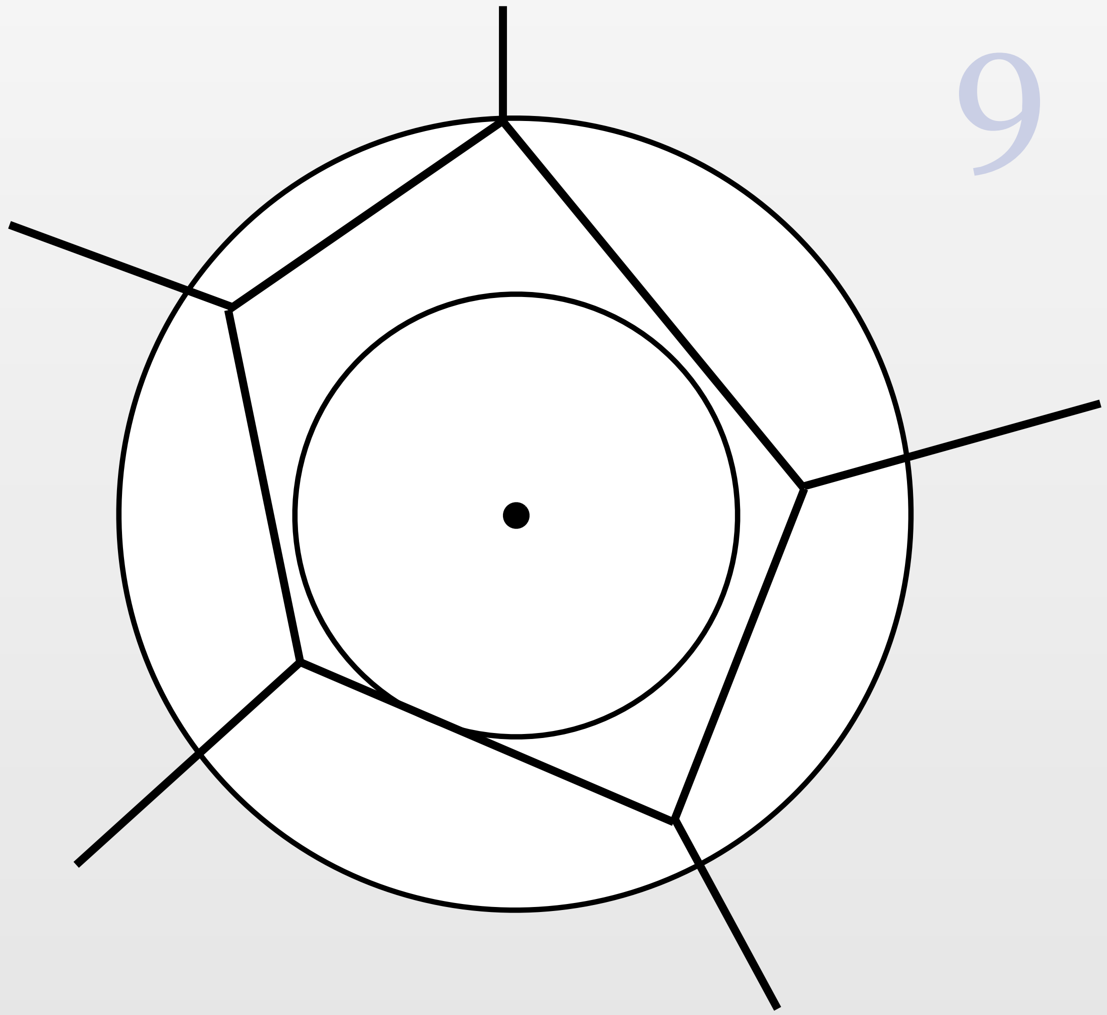
Delaunay balls have constant ply



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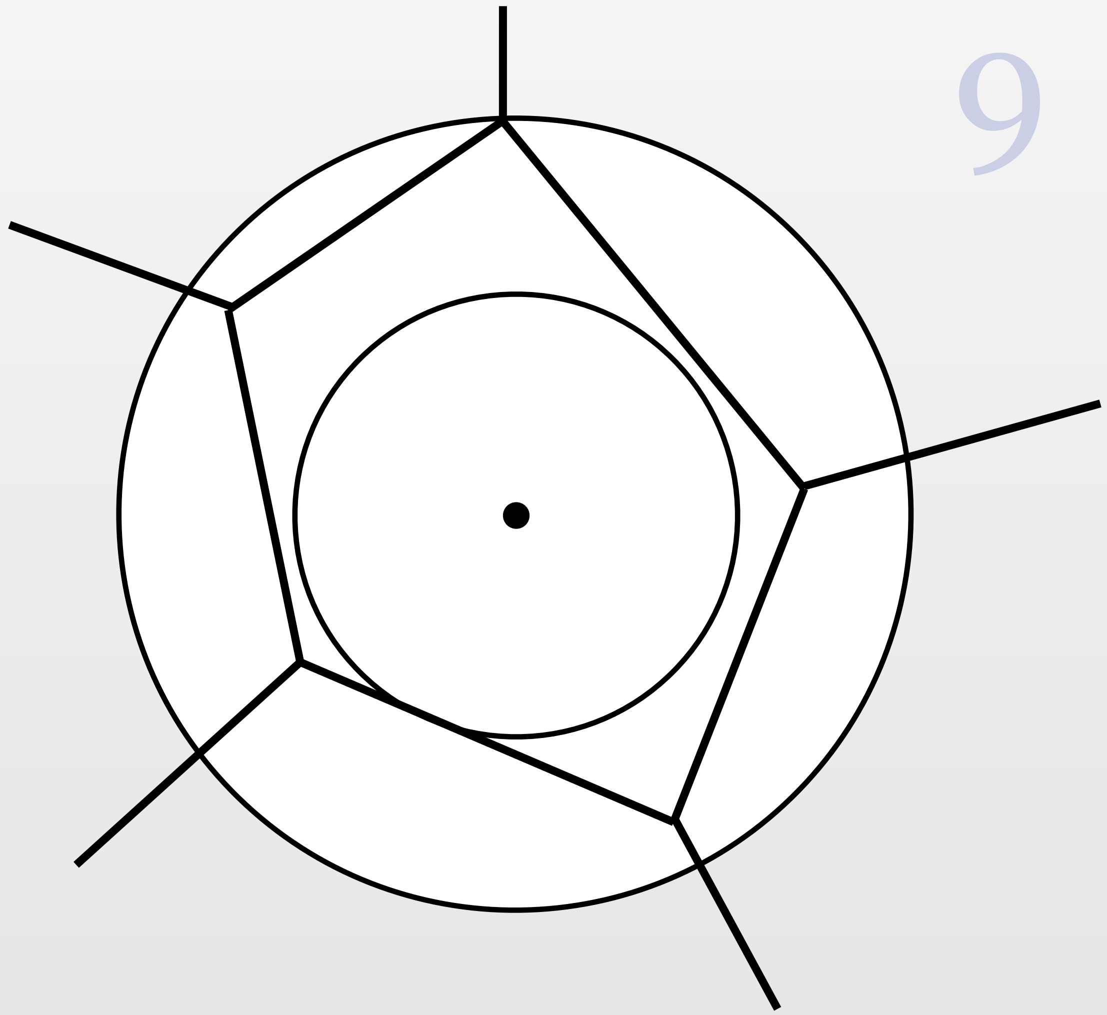

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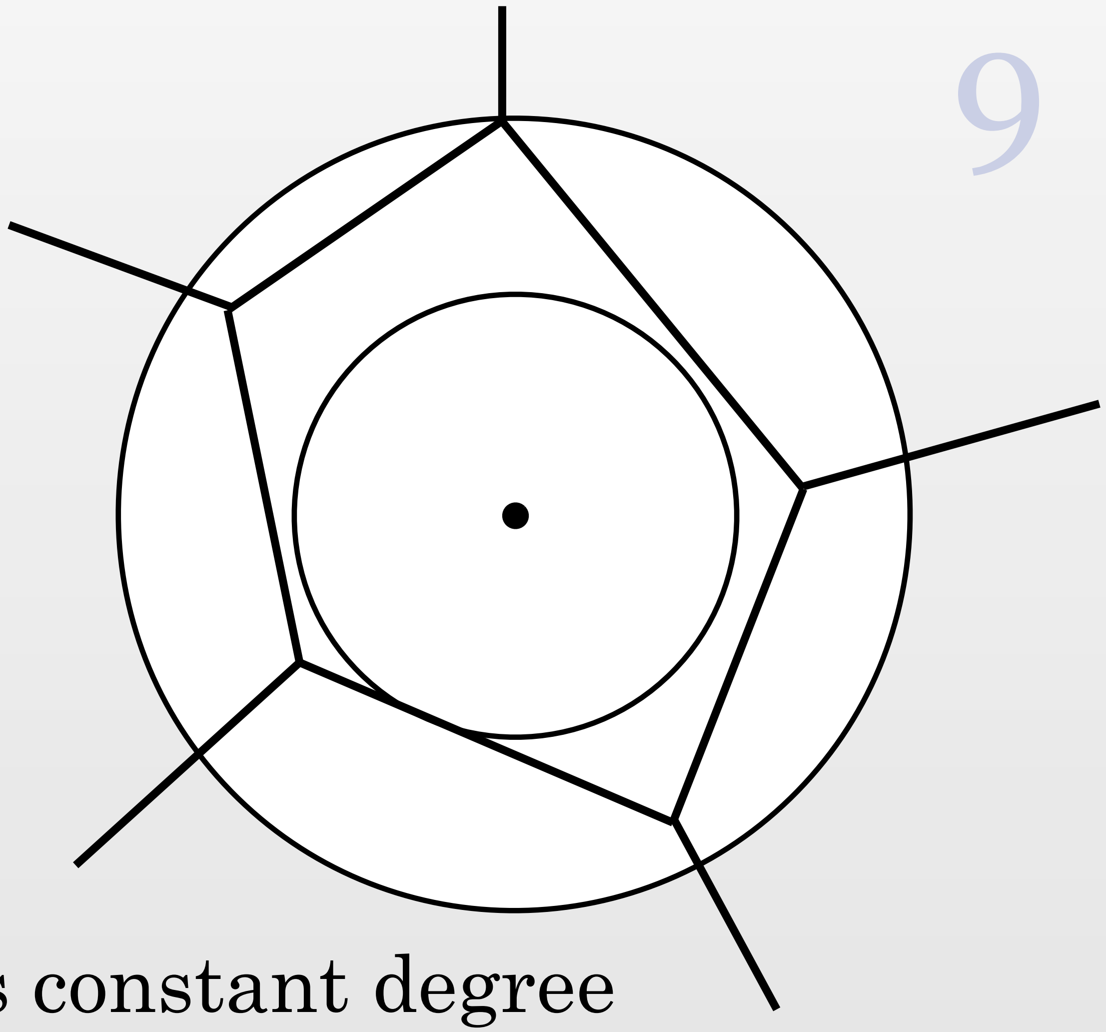

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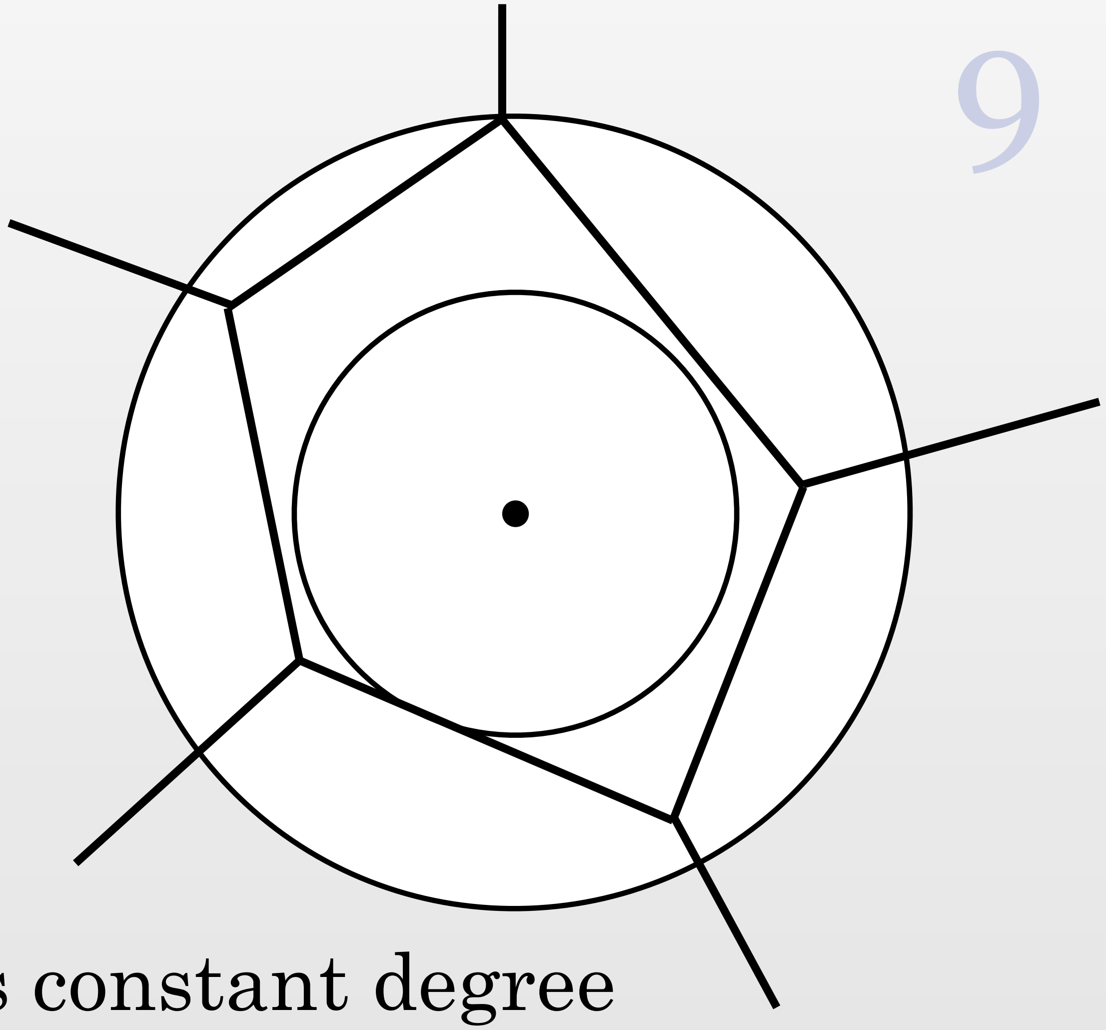

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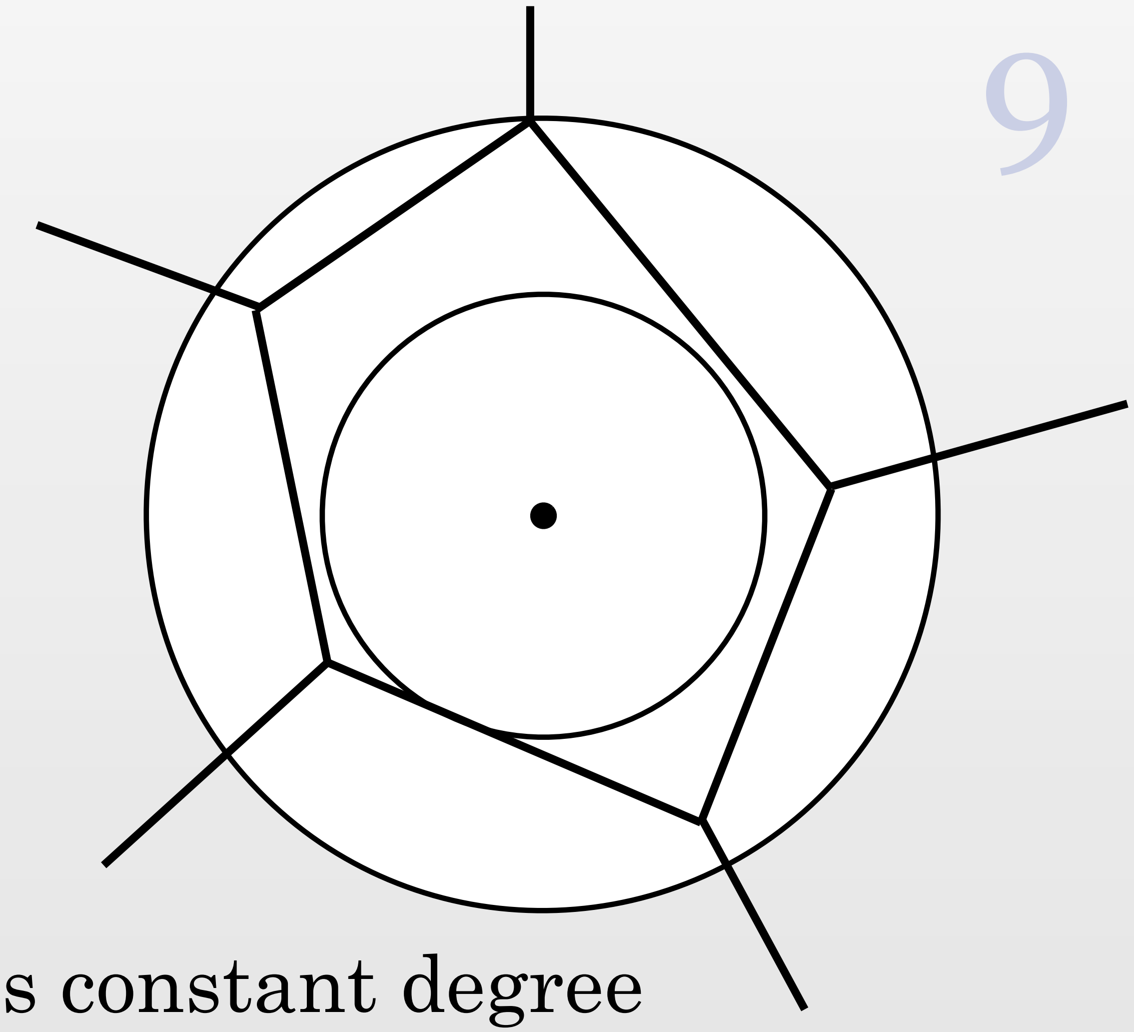

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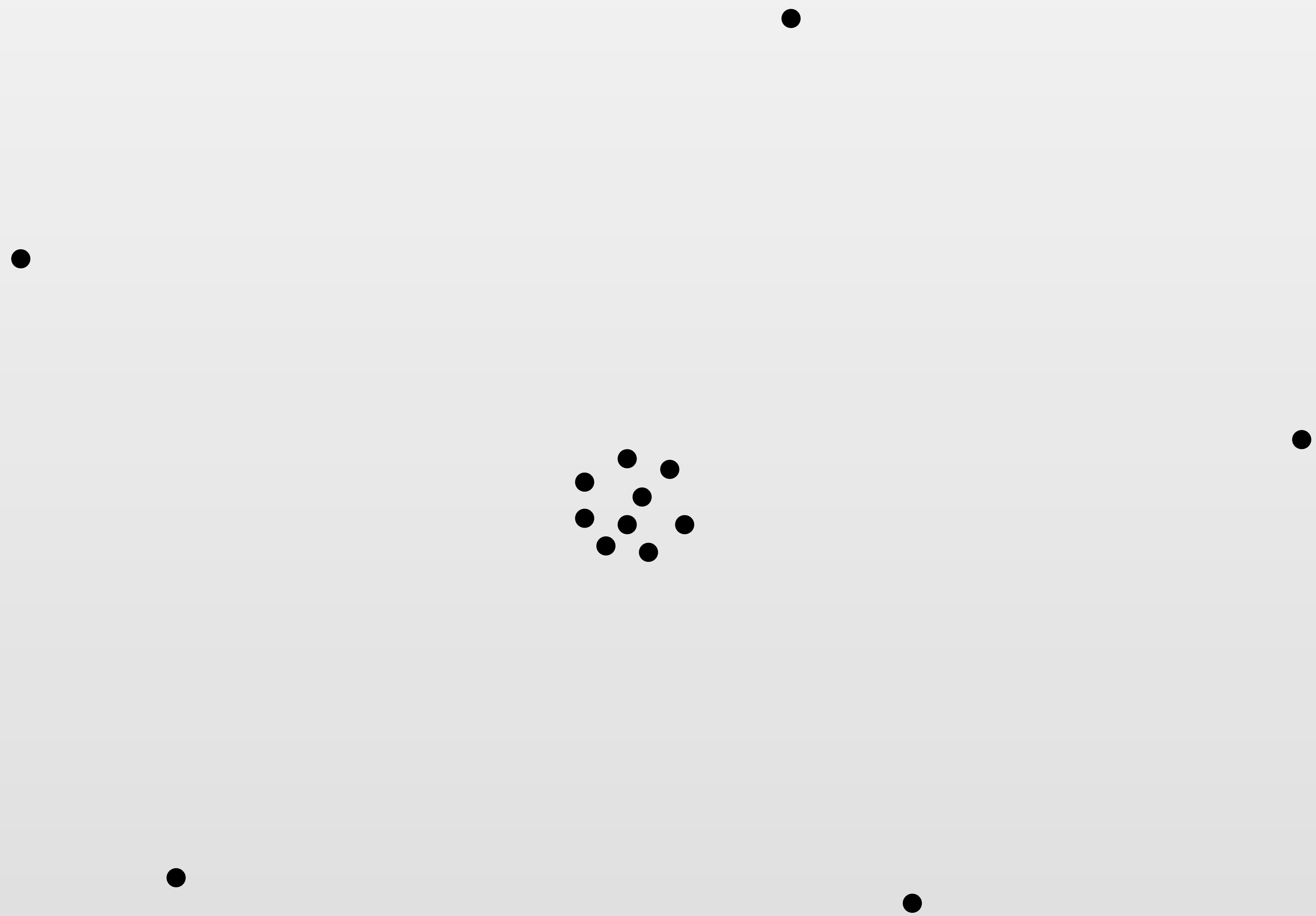
Voronoi Refinement: If some cell is skinny,
add a Steiner point at its farthest vertex.

We replace *quality* with *hierarchical quality*.



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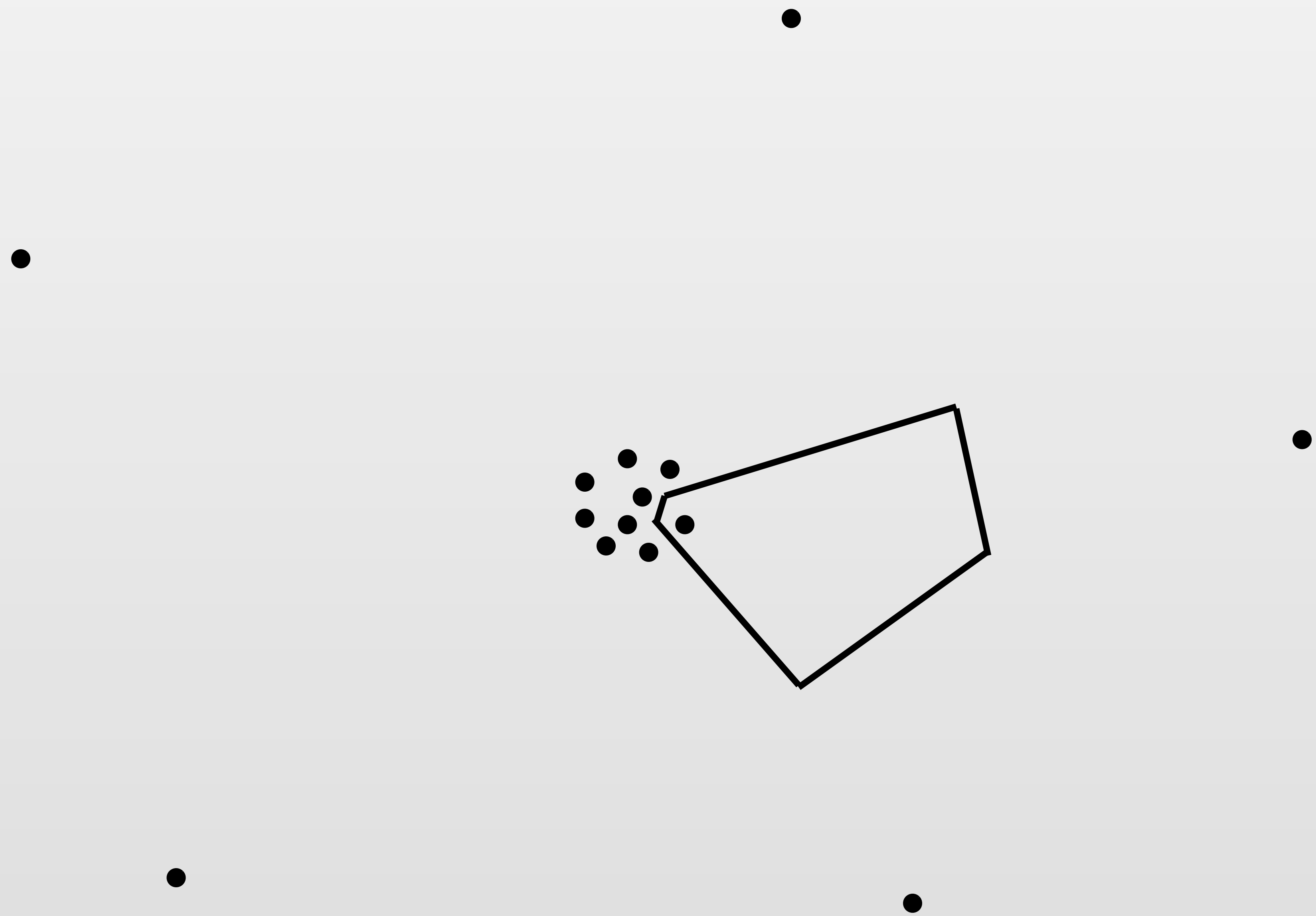

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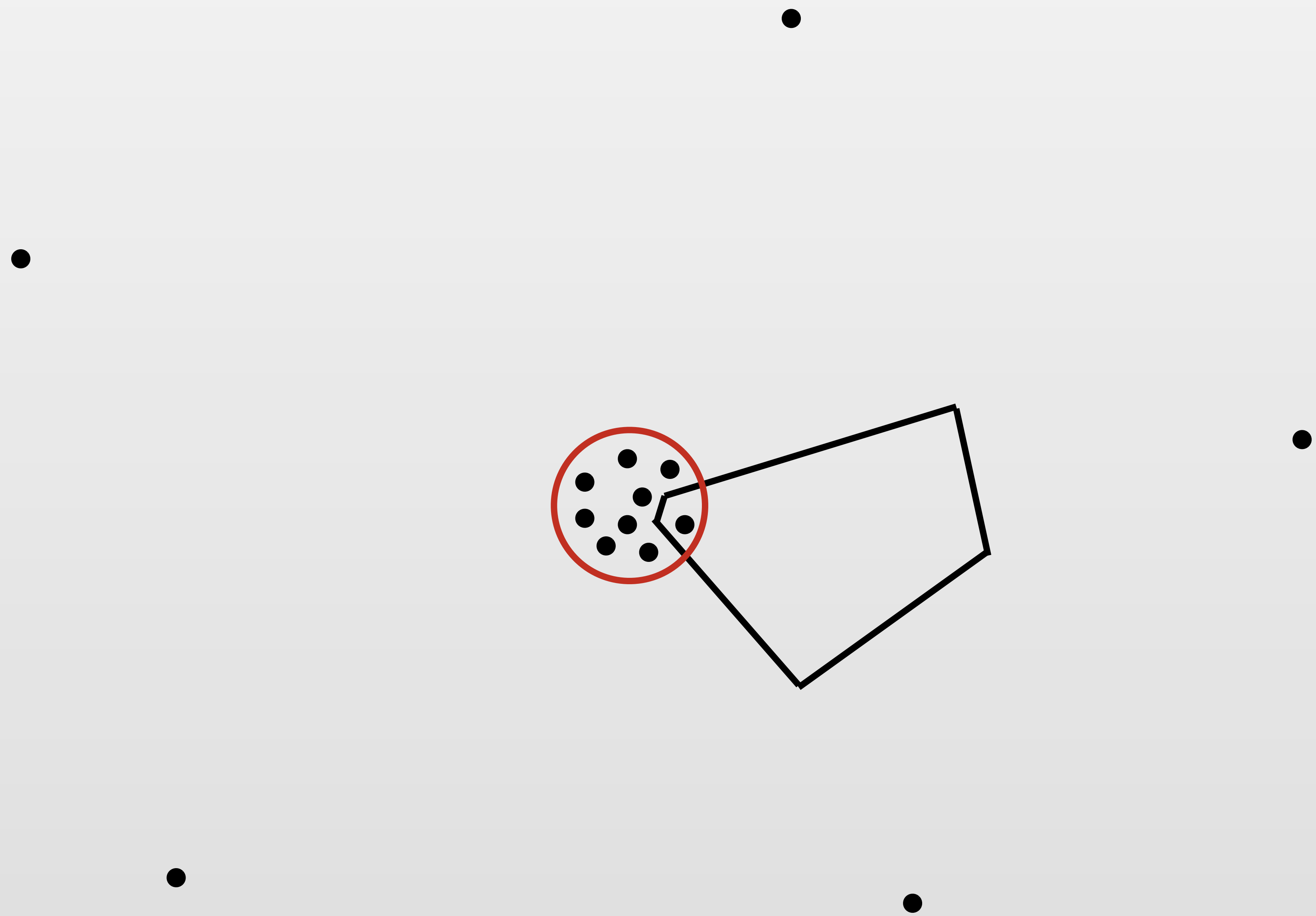

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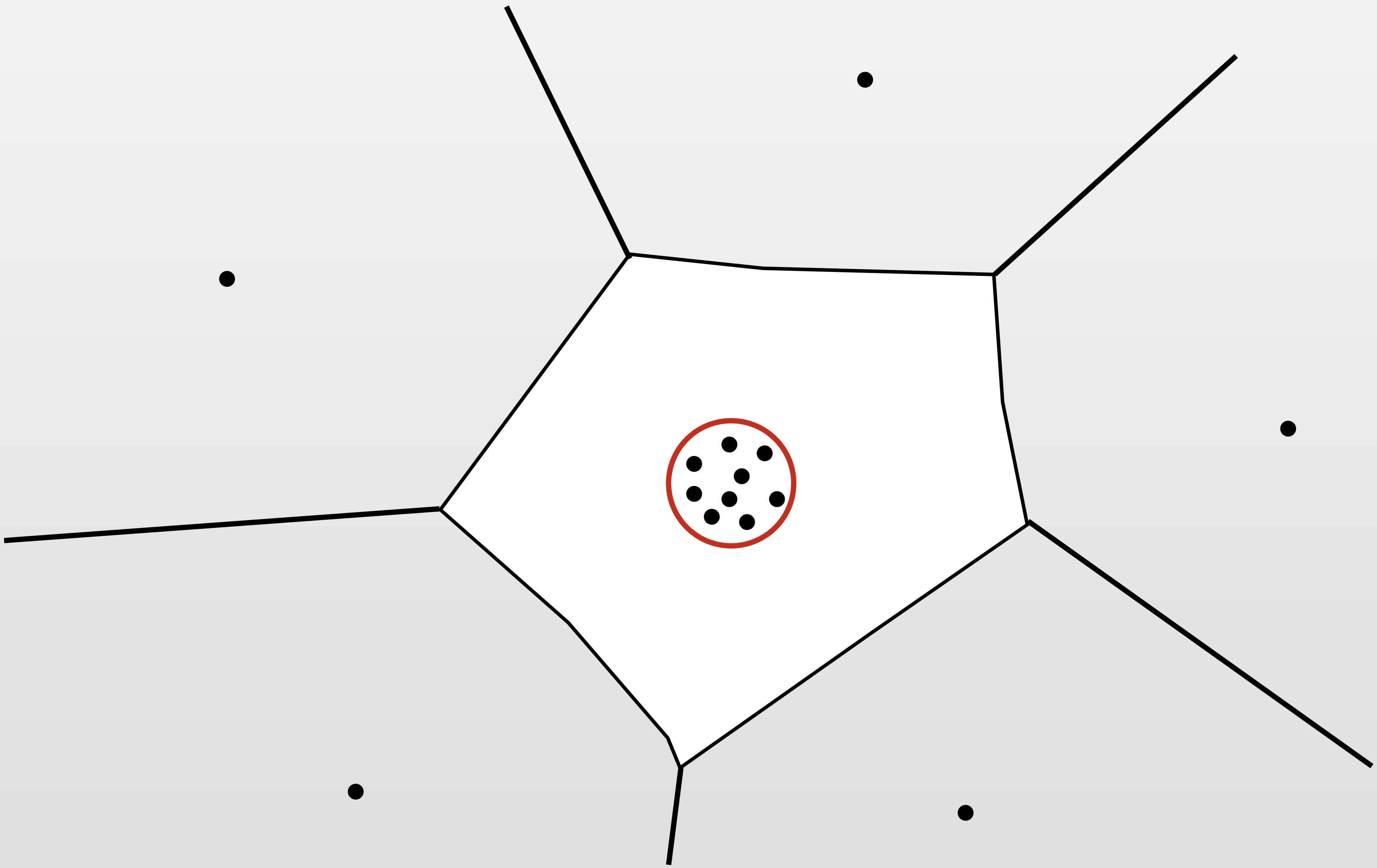

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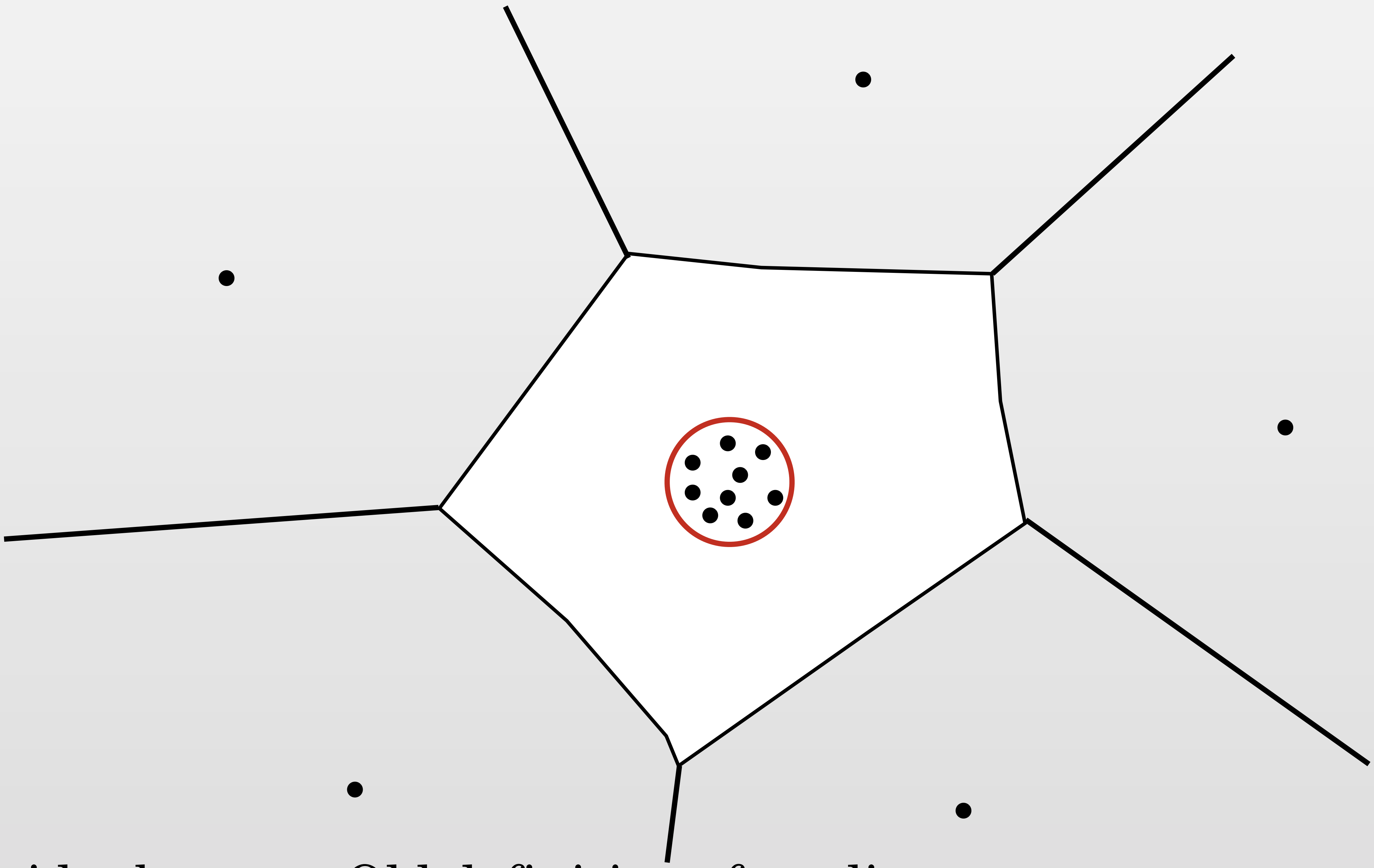

Inside the cage: Old definition of quality.

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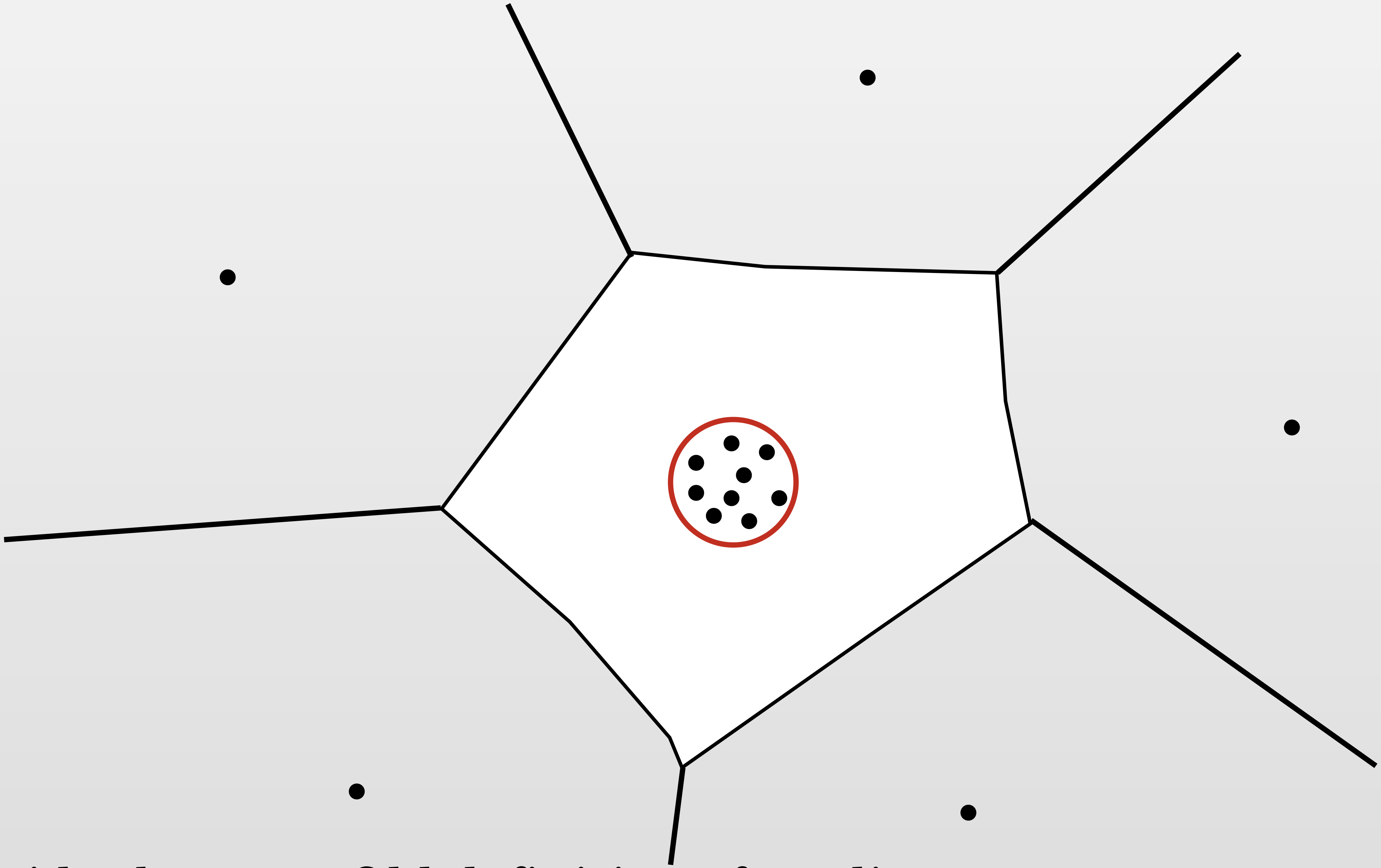

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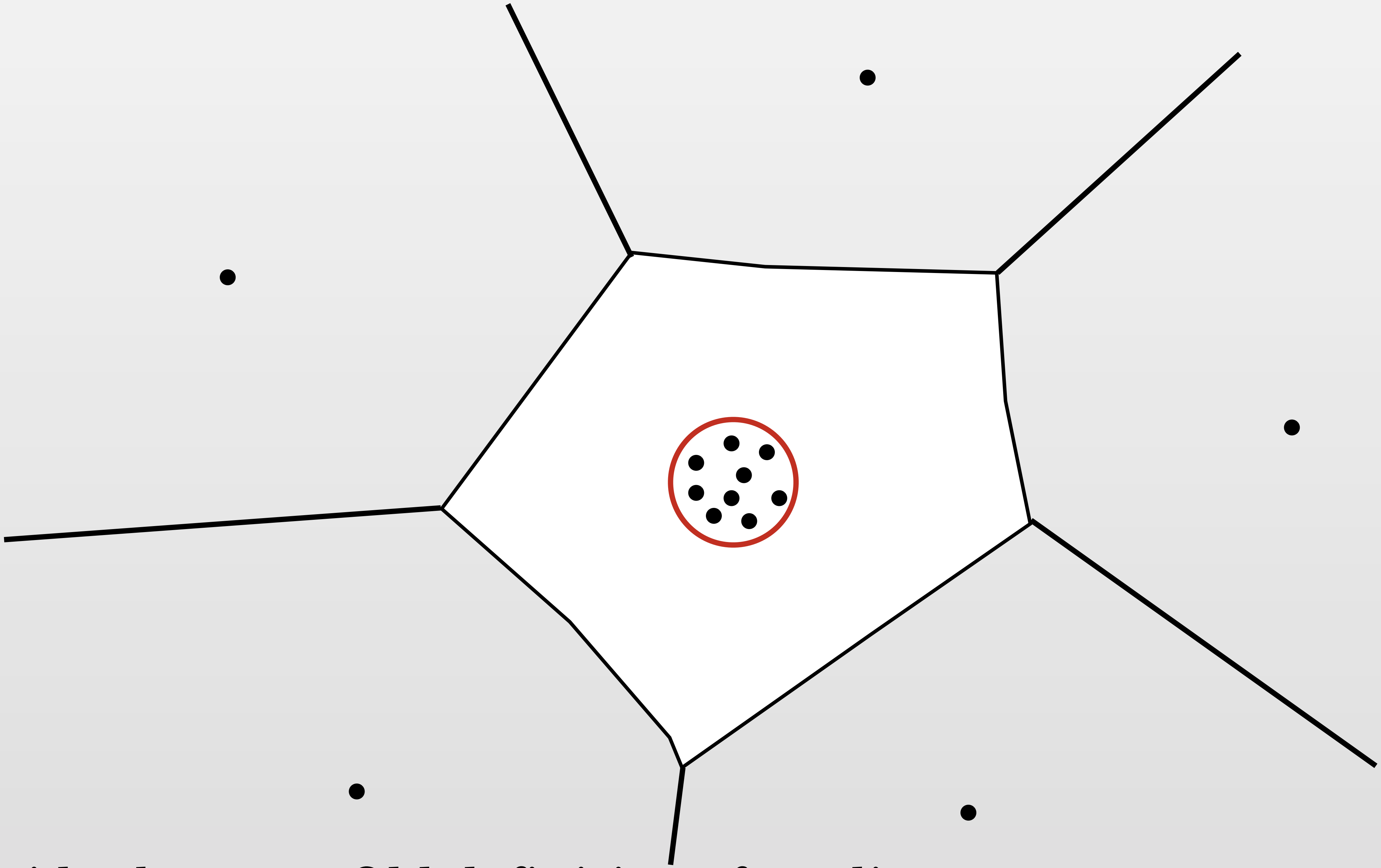

Inside the cage: Old definition of quality.
Outside: Treat the whole cage as a single object.

Has the same important properties as quality meshes: ply, degree,...

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This is how we avoid the worst-case Voronoi bounds.

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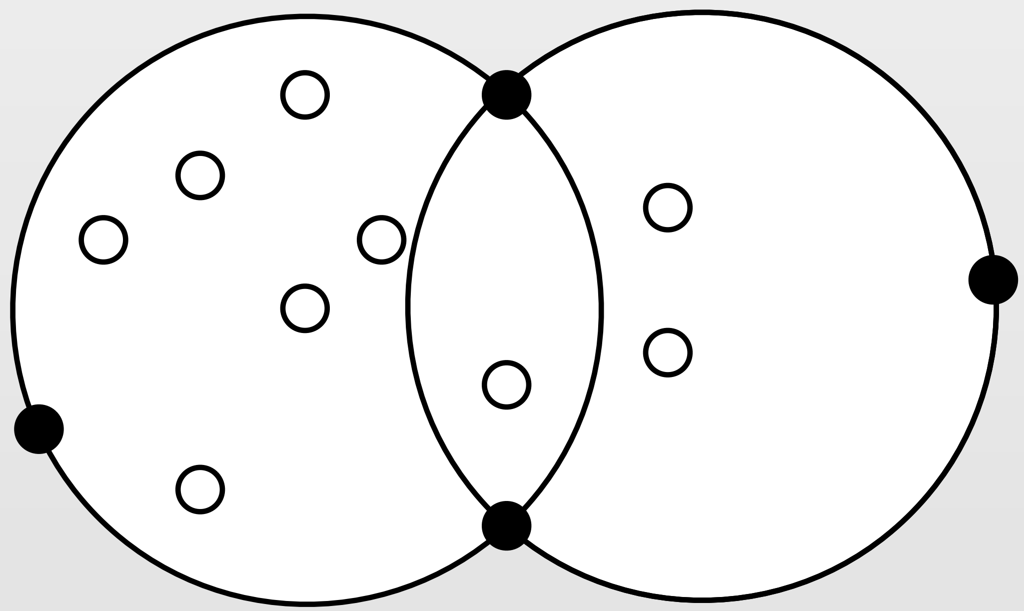
4. Store uninserted input points in the D-Balls.
5. Order the input points using range space nets.

Point Location in the D-Balls.

Idea: Store the uninserted points in the D-balls.
When the balls change, make local updates.



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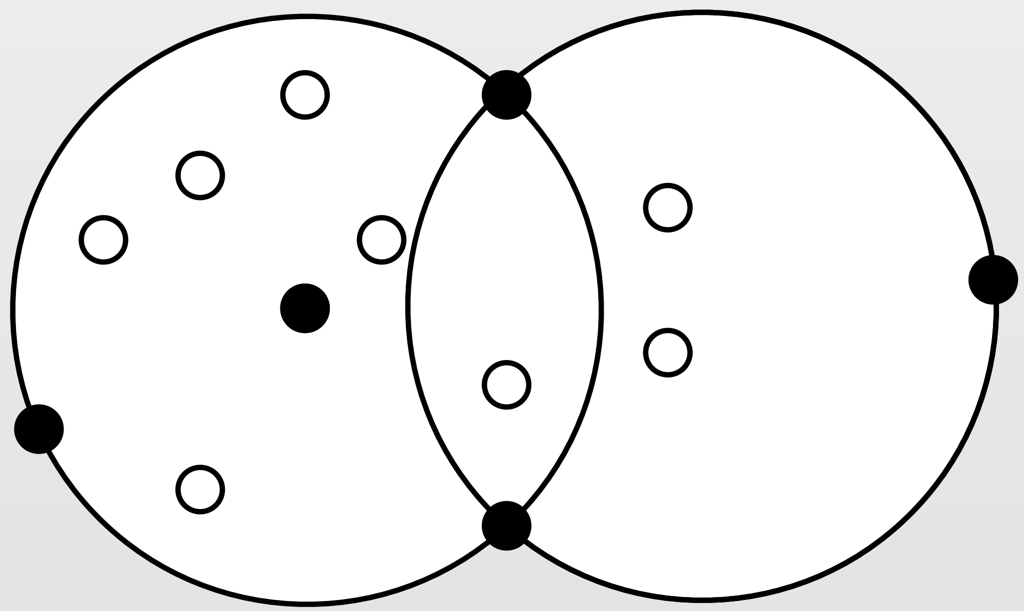

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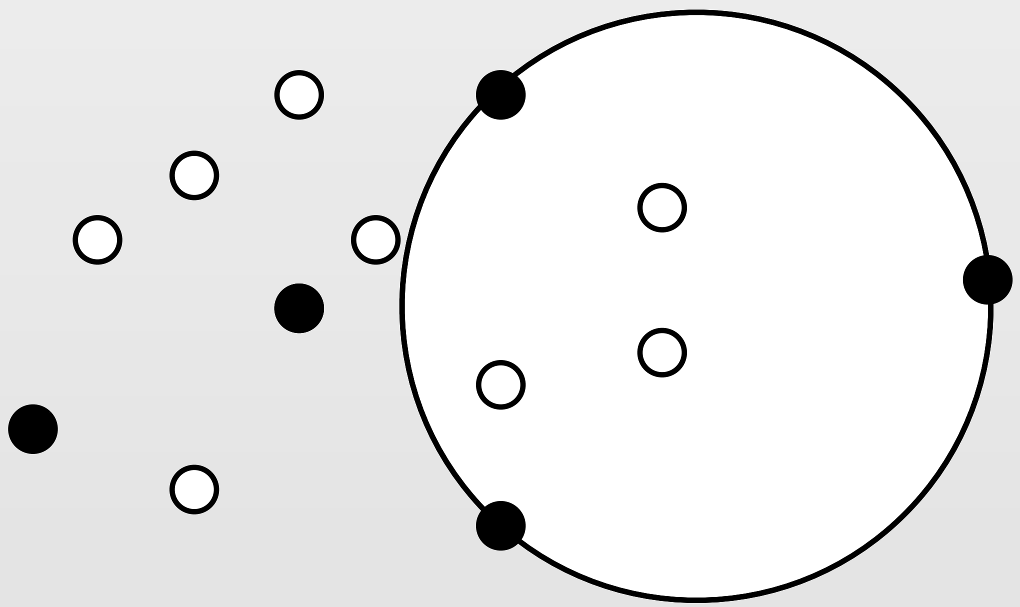

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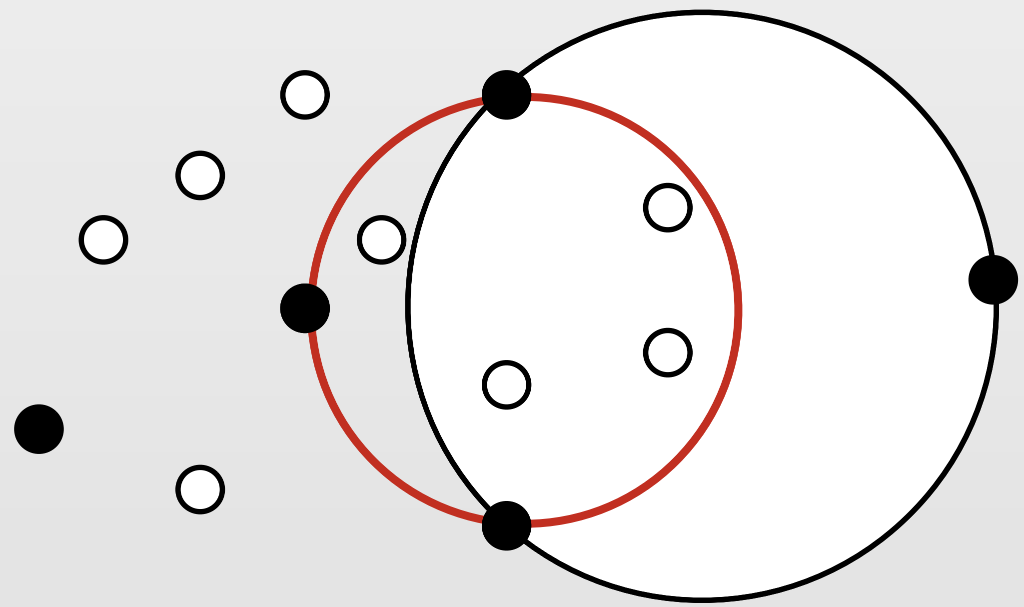

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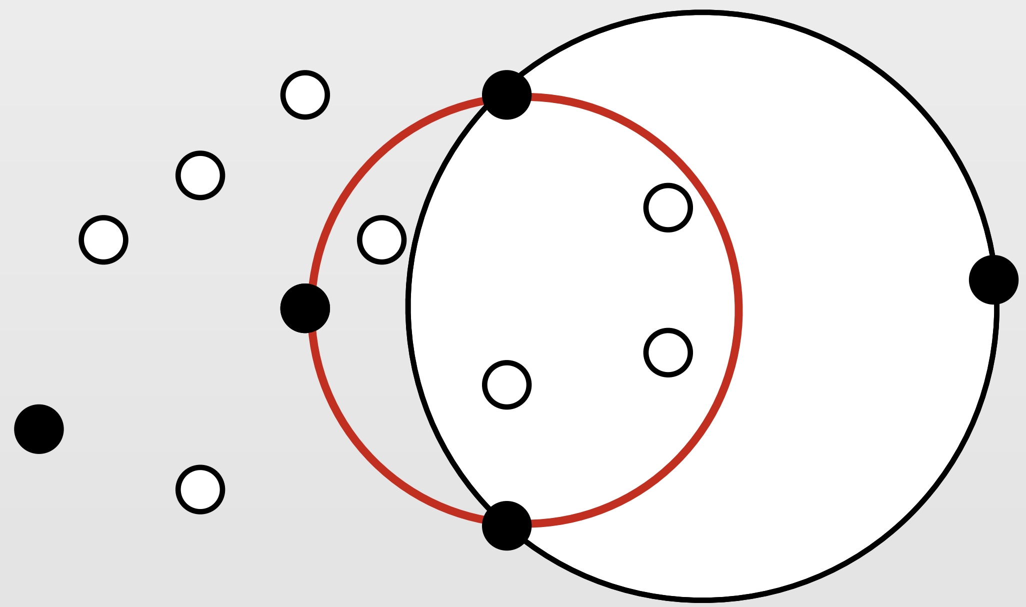

It's a history DAG!

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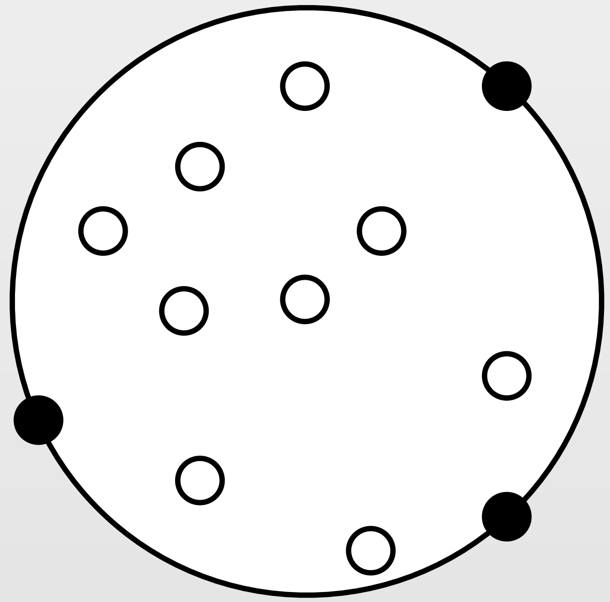
Theorem: [Chazelle & Matousek 96] For ε, d fixed constants, ε-nets of size $O(1)$ can be computed in $O(n)$ deterministic time.

Ordering the inputs

For each D-Ball, select a $\frac{1}{2d}$-net of the points it contains.
Take the union of these nets and call it a round.
Insert these.
Repeat.



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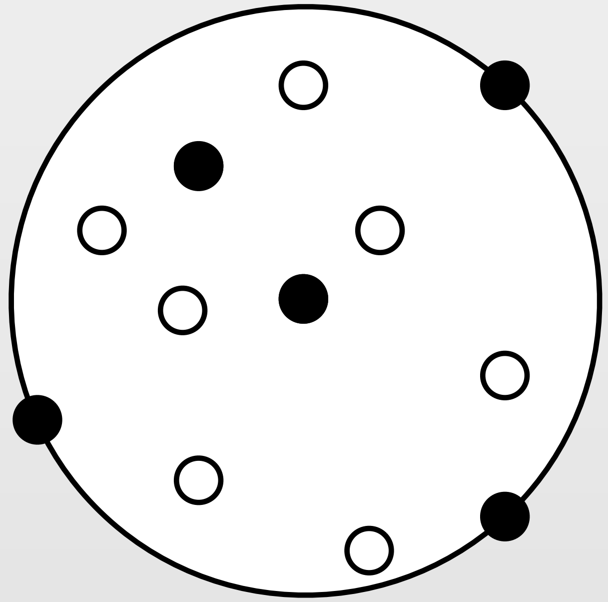

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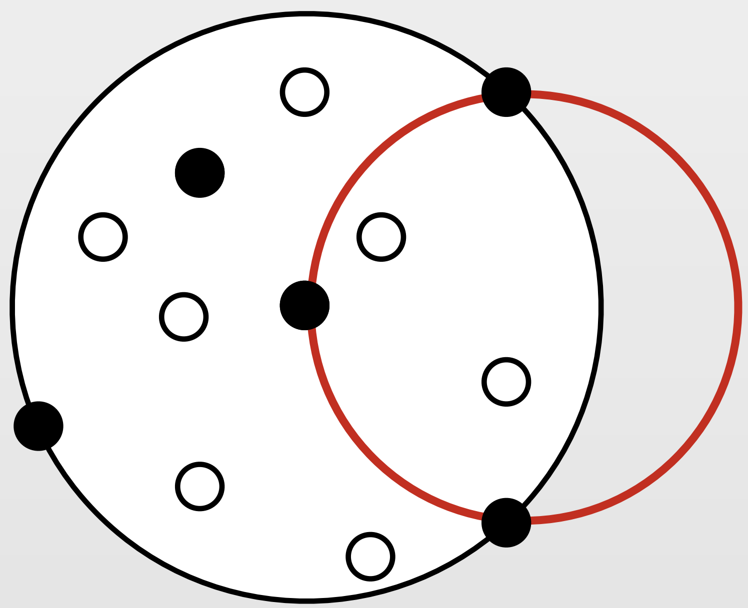

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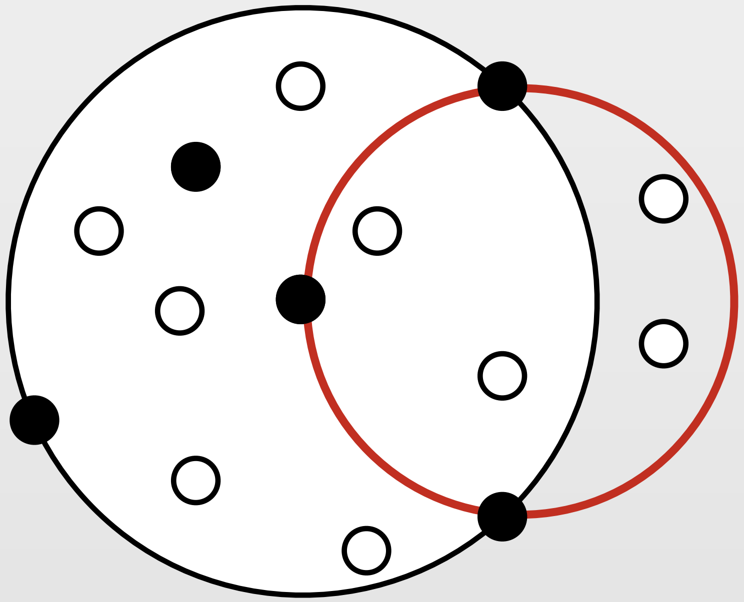
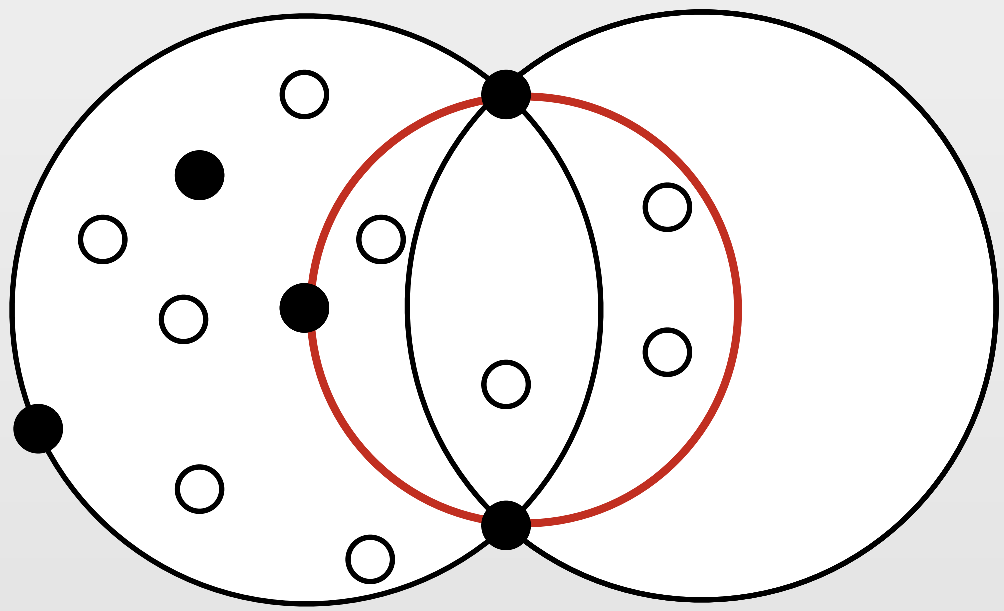


Ordering the inputs

For each D-Ball, select a $\frac{1}{2d}$-net of the points it contains.
Take the union of these nets and call it a round.
Insert these.
Repeat.



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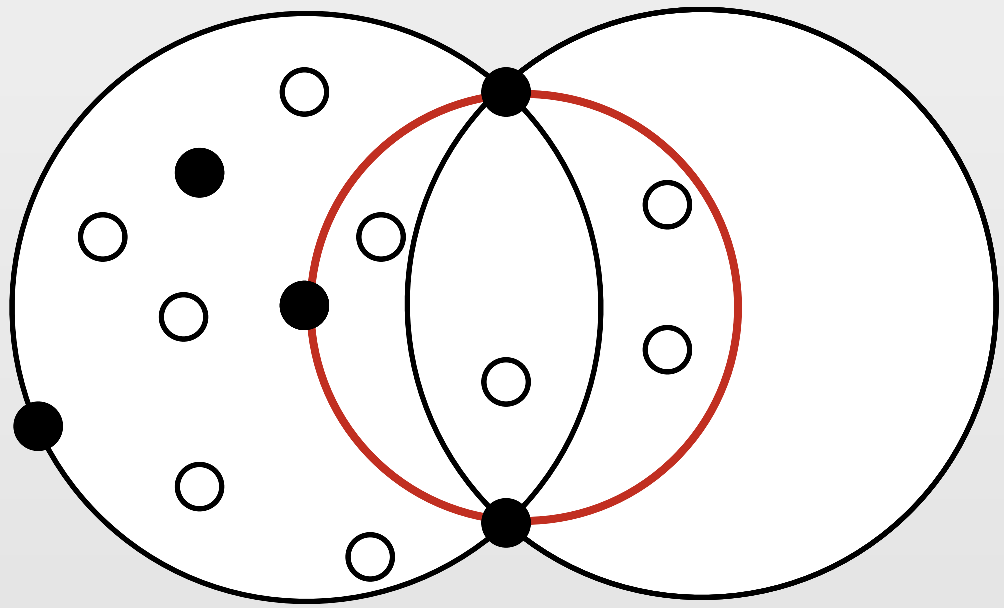

Lemma. *Let M be a set of vertices. If an open ball B contains no points of M, then B is contained in the union of d D-balls of M.*

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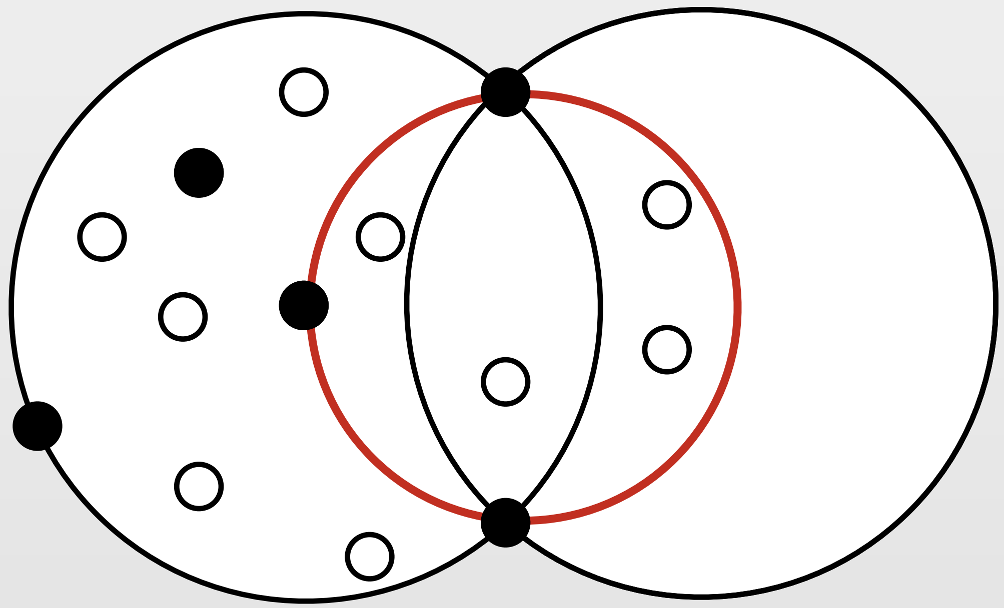

$$\max_{\text{D-balls } B} |B \cap P|$$

Goes down by a factor of 2 each round.

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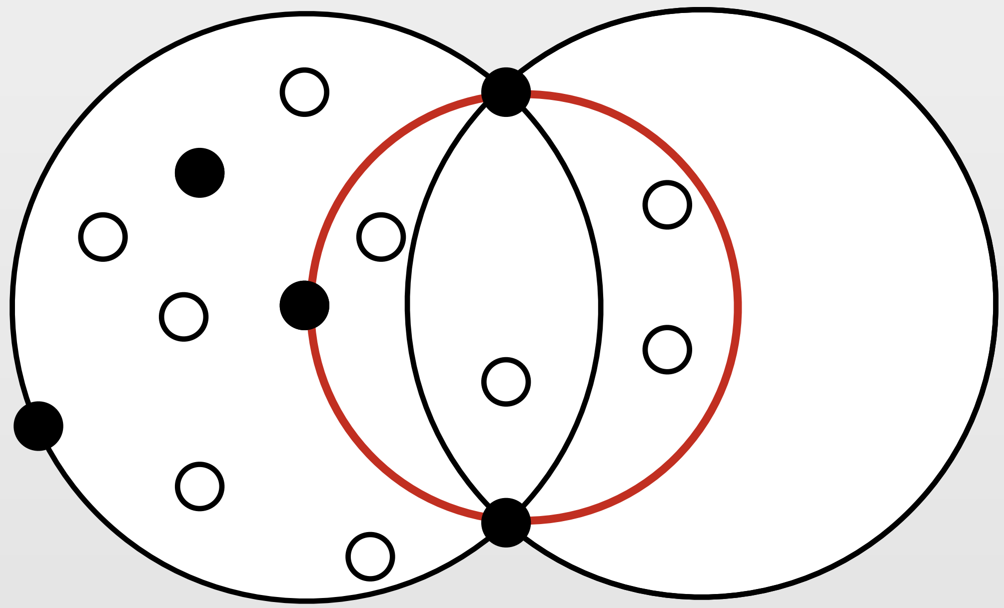

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=> $\log(n)$ Rounds

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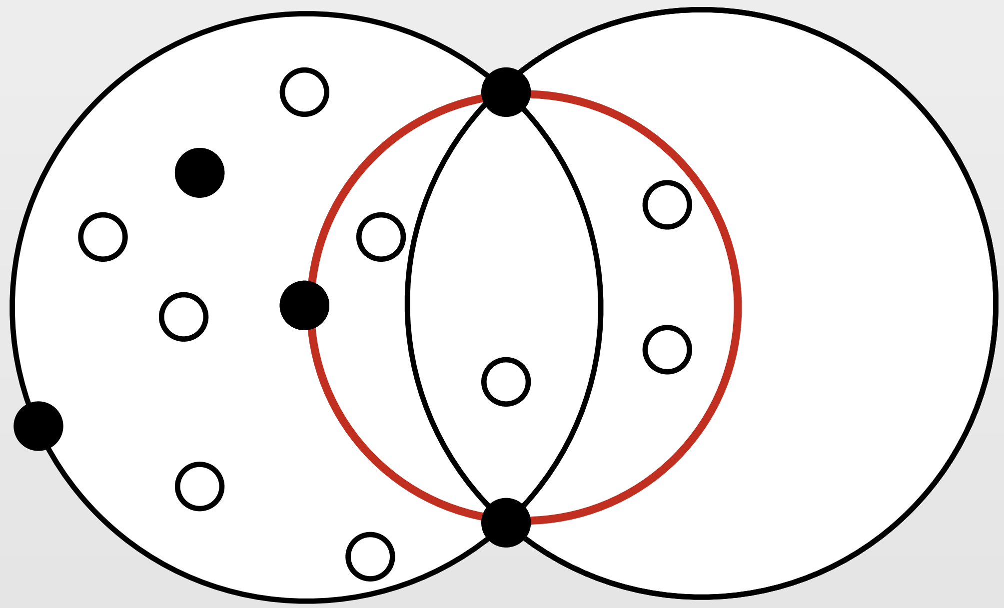

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Note: After $k = \log \frac{1}{\epsilon}$ rounds, the intermediate mesh is a **weak ϵ-net** for the range space of Euclidean balls.
Size: $O\left(\frac{1}{\epsilon}\right)$, Time: $O(nk) = O(n \log \frac{1}{\epsilon})$.

To complete the analysis, we must show that the cost of a Round is O(n).

log n rounds x $O(n)$ time/round = $O(n \log n)$

Watch an uninserted point x.
Claim: x only gets touched $O(1)$ times per round.

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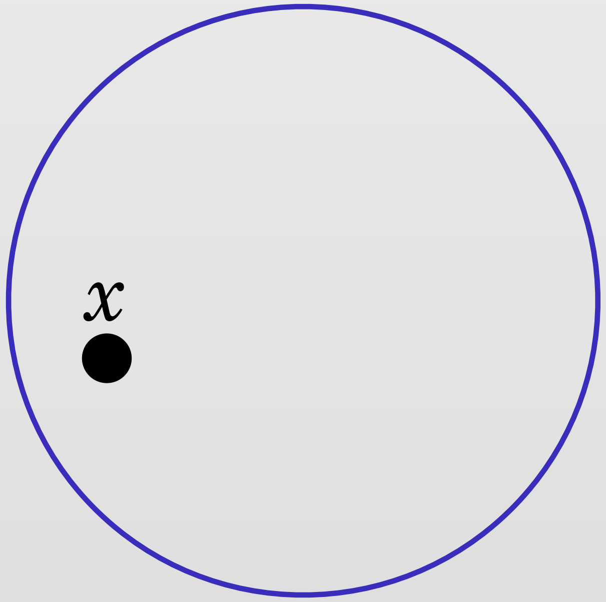

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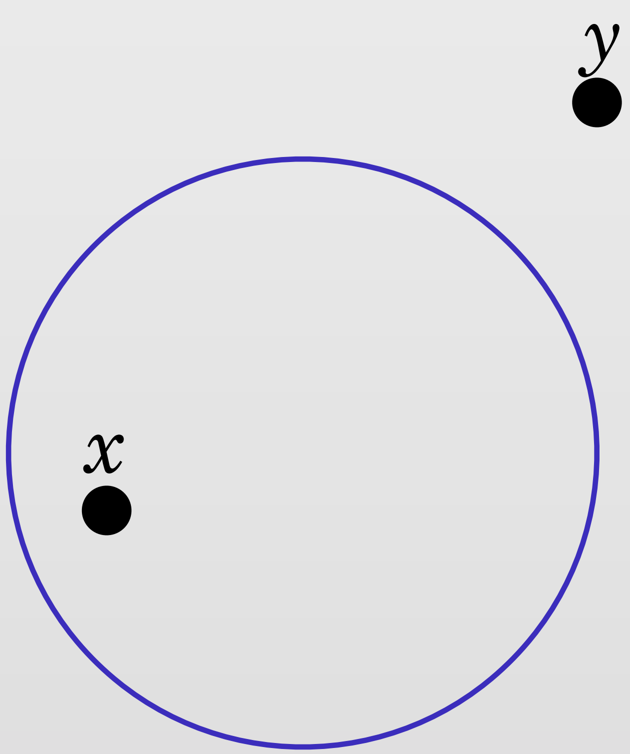

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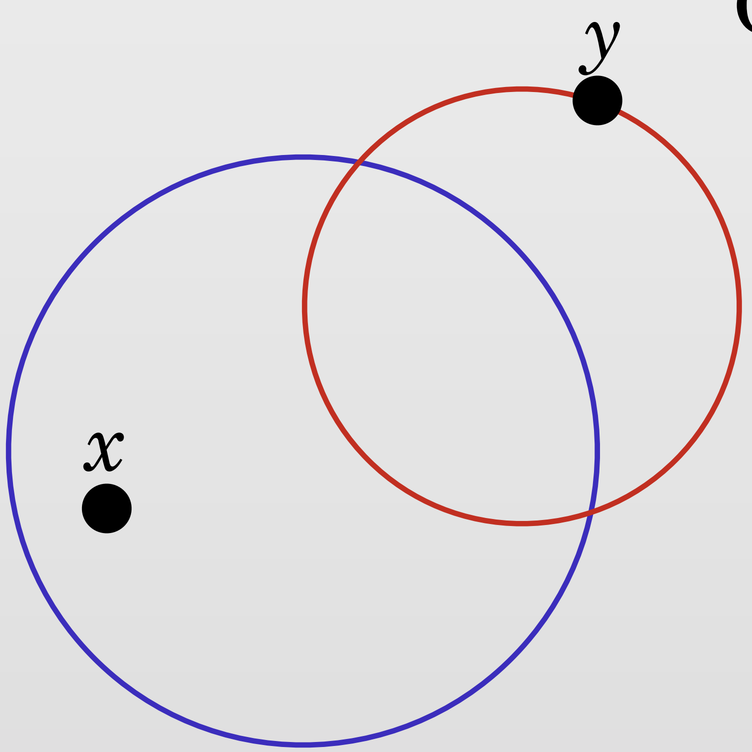

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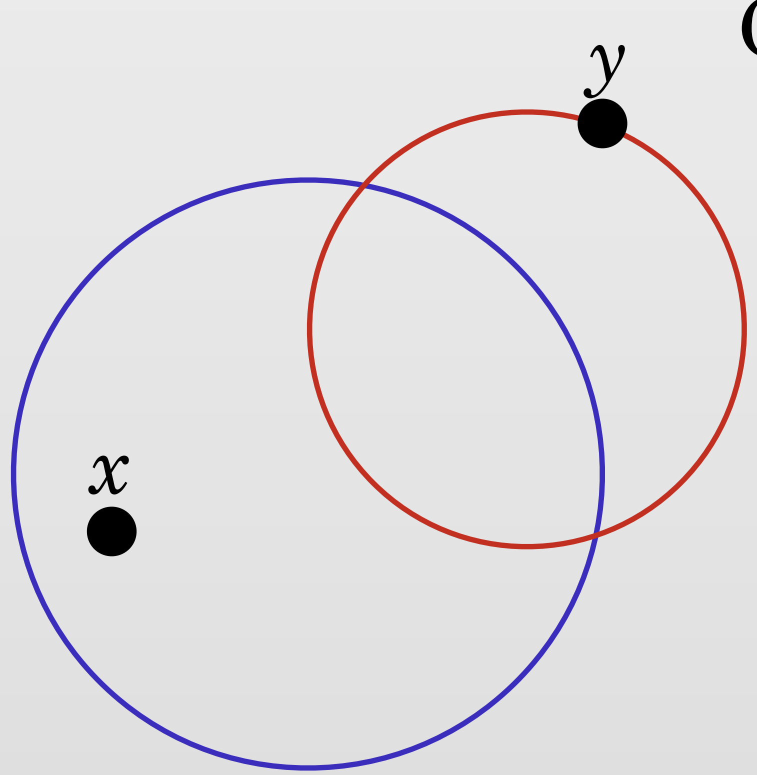

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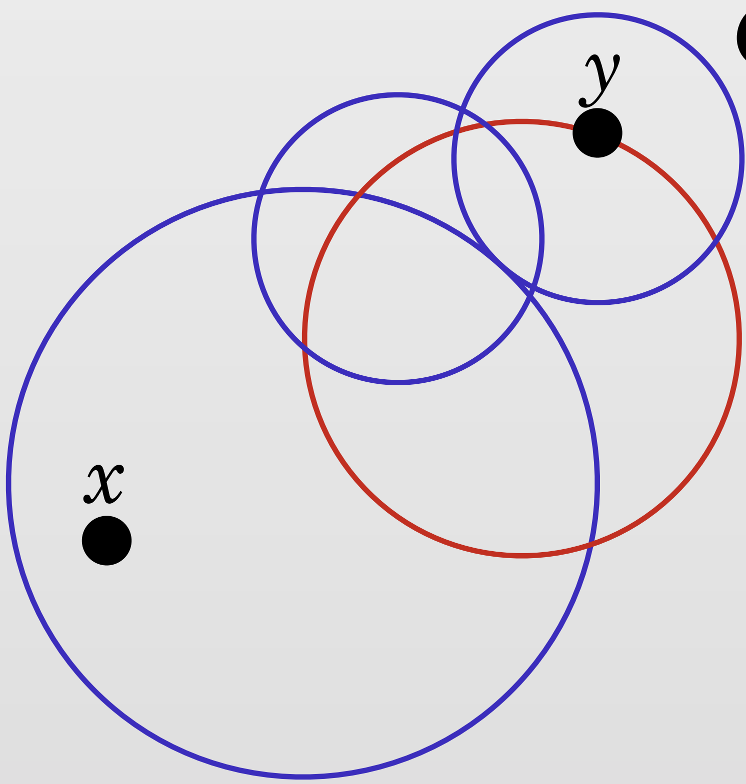

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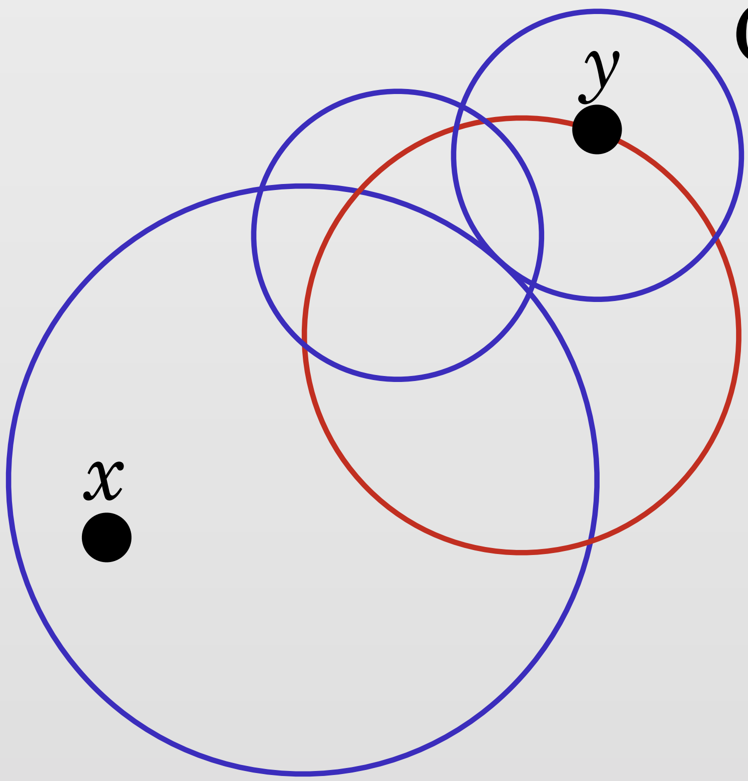

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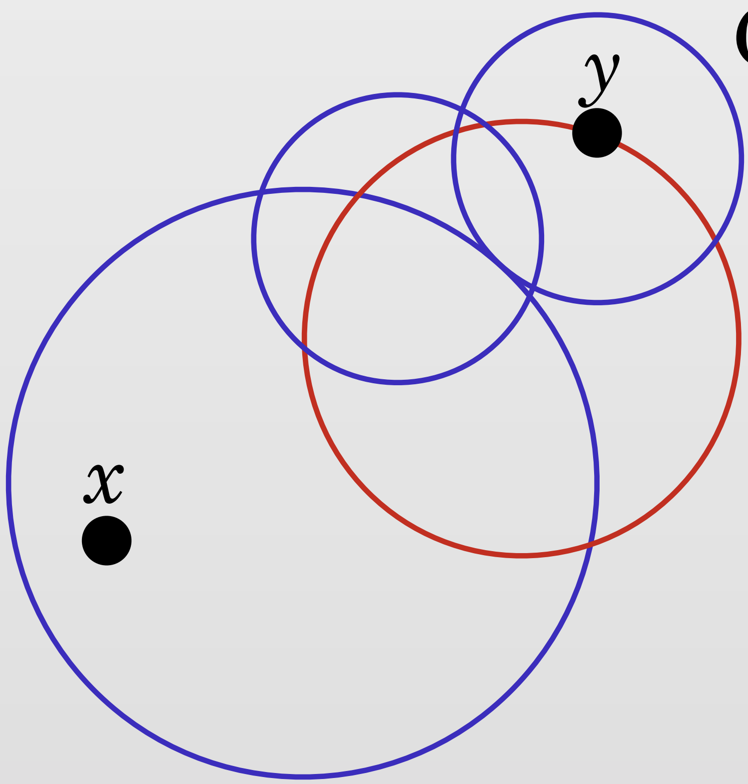

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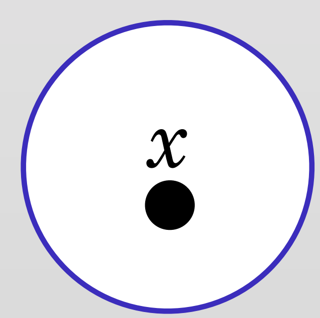

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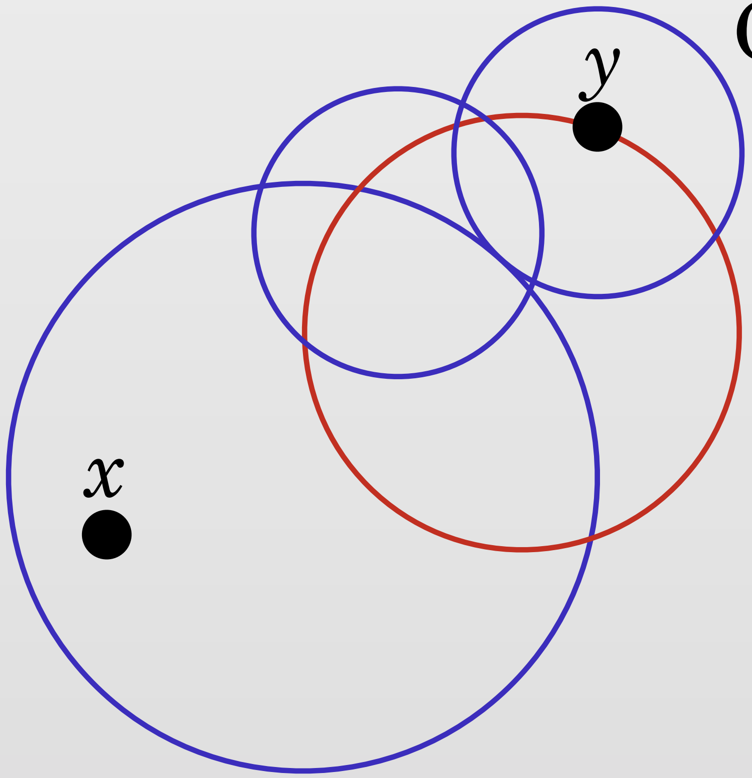

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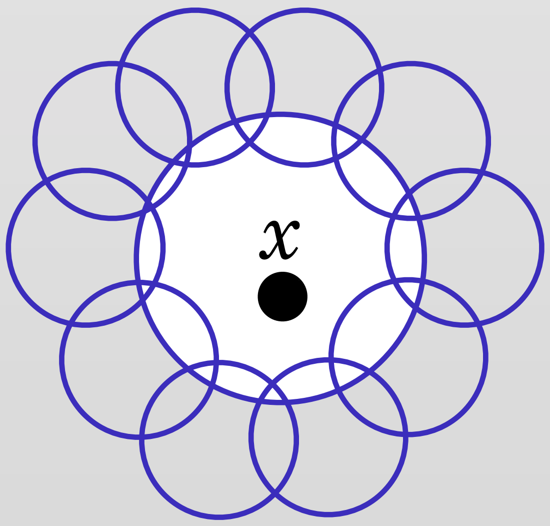

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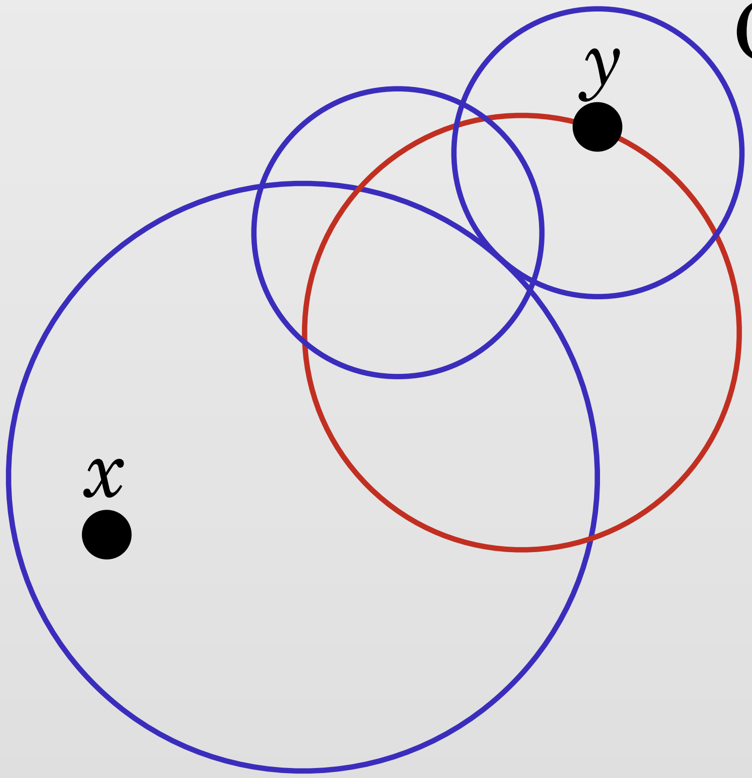

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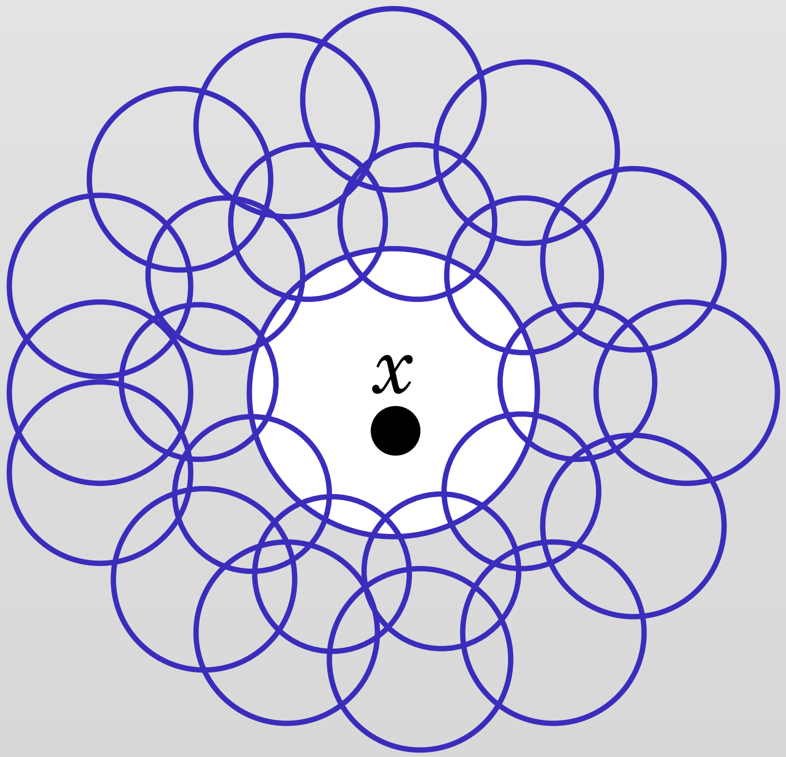

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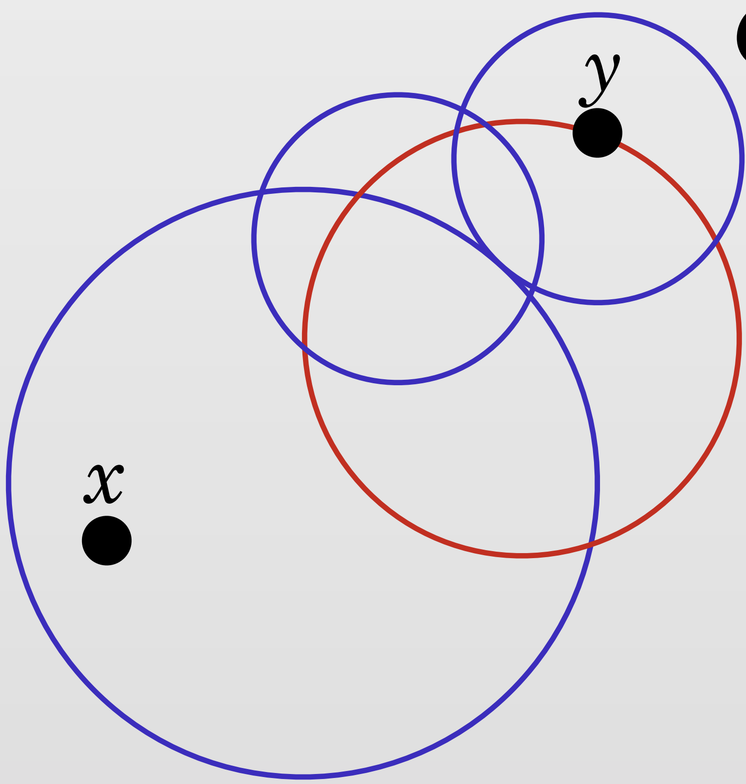

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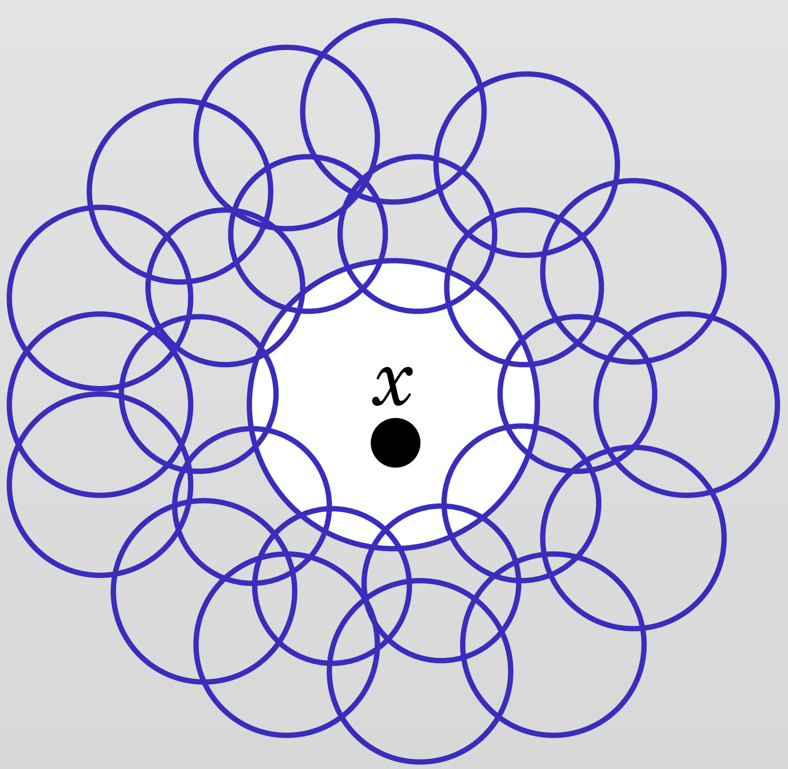

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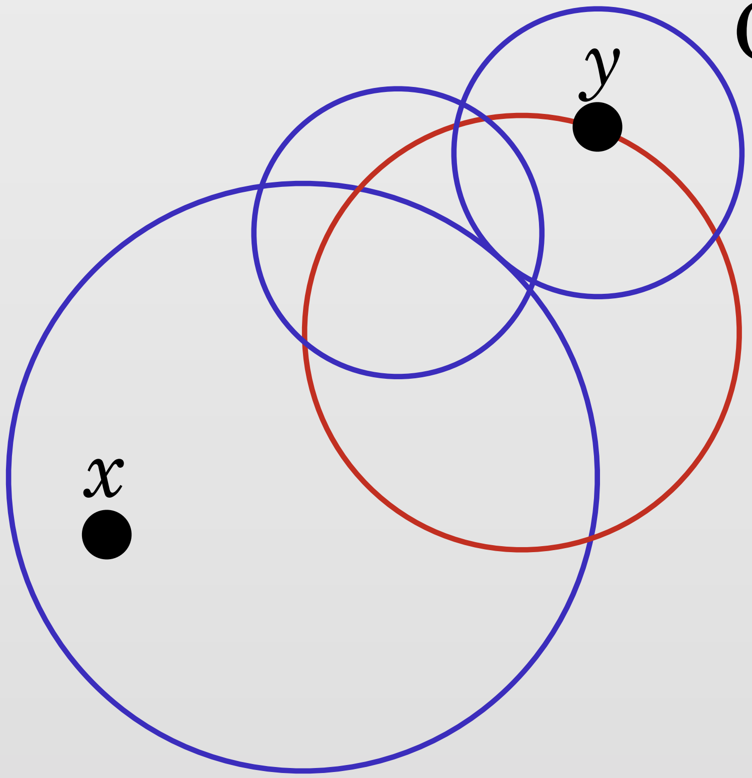

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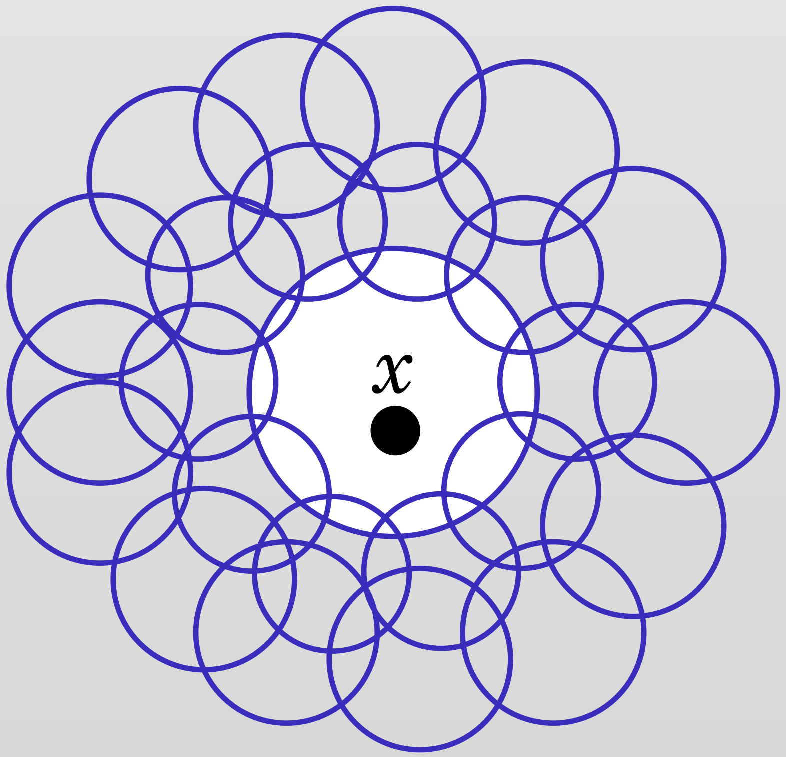

$O(n)$ total work per round.

Meshing Points in (optimal)
$O(n \log n + m)$ time.

Thank you.