Beating the Spread: Time-Optimal Point Meshing

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Meshing Points

Input: $P \subset \mathbb{R}^d$
Output: $M \supset P$ with a “nice” Voronoi diagram

$n = |P|, m = |M|$
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\[
\tau \geq 2 + \varepsilon
\]
Prior Work
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Delaunay Refinement:
Prior Work

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Chew ‘89 2D
Prior Work

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Chew ‘89 ................................................................. 2D
Ruppert ’95 ............................................................ Optimality in 2D
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- Hudson - Miller - Phillips ‘06

2D Optimality
- Ruppert in 2D
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SVR O(n log Δ + m)
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- Bern - Eppstein - Gilbert ’94 ................................. QT meshing
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Our Result

$O(n \log n + m)$

for point sets in $d$ dimensions
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Our Result
$O(n \log n + m)$ for point sets in d dimensions
Hides dimension terms
Beating the spread.
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Let \( s = |x - y| \) where \((x, y) \in \binom{P}{2}\) is the closest pair.
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- Point location
- Output sensitive
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Output sensitive

For some point sets, \( m = \Omega(n \log \Delta) \)
Complexity: How big is the mesh?

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How many Steiner Points?

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\[ m = O(n) \] as long as there are no big empty annuli with 2 or more points inside [MPS08].
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How many simplices?

Only \( O(m) \) simplices.
Compare to \( m^{\lfloor d/2 \rfloor} \) for general Delaunay triangulations. Constants depend on aspect ratio.
Complexity: How hard is it to compute a mesh?
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$O(n \log n + m)$
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\[ O(n \log n) \]

Compute a **hierarchical quality** mesh.
Complexity: How hard is it to compute a mesh?

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Compute a \textit{hierarchical quality} mesh.

Key fact: Size is \( O(n) \)
Complexity: How hard is it to compute a mesh?

\[ O(n \log n + m) \]

Point Location \quad Output Sensitive

This is optimal in the comparison model.

\[ O(n \log n) + O(m) \]

Compute a *hierarchical quality* mesh.

**Key fact:** Size is \( O(n) \)
Complexity: How hard is it to compute a mesh?

\[ O(n \log n + m) \]

Point Location  
Output Sensitive

This is optimal in the comparison model.

\[ O(n \log n) + O(m) \]

Compute a \textit{hierarchical quality} mesh.  
Finishing post-process. (easy)

\textbf{Key fact: Size is } \textit{O(n)}
The Delaunay Triangulation is the dual of the Voronoi Diagram.
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Delaunay balls have constant ply
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Total number of faces is $O(m)$
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Insertions only take constant time
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**Voronoi Refinement:** If some cell is skinny, add a Steiner point at its farthest vertex.
We replace *quality* with *hierarchical quality*.
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Inside the cage: Old definition of quality.
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Inside the cage: Old definition of quality.
Outside: Treat the whole cage as a single object.
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Inside the cage: Old definition of quality.
Outside: Treat the whole cage as a single object.

Has the same important properties as quality meshes: ply, degree,...
The sparse meshing model:
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1. Build a Voronoi diagram incrementally.
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This is how we avoid the worst-case Voronoi bounds.
The New Algorithm

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4. Store uninserted input points in the D-Balls.
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1. Build a Voronoi diagram incrementally.
2. Interleave input and Steiner point insertions.
3. Recover quality after each input point.
   - hierarchical quality
4. Store uninserted input points in the D-Balls.
5. Order the input points using range space nets.
Point Location in the D-Balls.

Idea: Store the uninserted points in the D-balls. When the balls change, make local updates.
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It’s a history DAG!
Range Nets

**Definition.** A *range space* is a pair \((X, R)\), where \(X\) is a set (the vertices) and \(R\) is a collection of subsets (the ranges).
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For us \(X = P\) and \(R\) is the set of open balls.
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**Definition.** Given a range space \((X, R)\), a set \(N \subset X\) is a *range space \(\varepsilon\)-net* if for all ranges \(r \in R\) that contain at least \(\varepsilon |X|\) vertices, \(r\) contains a vertex from \(N\).
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**Theorem:** [Chazelle & Matousek 96] For \(\varepsilon, d\) fixed constants, \(\varepsilon\)-nets of size \(O(1)\) can be computed in \(O(n)\) deterministic time.
Ordering the inputs

For each D-Ball, select a $\frac{1}{2d}$-net of the points it contains. Take the union of these nets and call it a round. Insert these. Repeat.
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**Lemma.** Let \( M \) be a set of vertices. If an open ball \( B \) contains no points of \( M \), then \( B \) is contained in the union of \( d \) D-balls of \( M \).
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\[ \max_{\text{D-balls } B} |B \cap P| \]

Goes down by a factor of 2 each round.
Lemma. Let $M$ be a set of vertices. If an open ball $B$ contains no points of $M$, then $B$ is contained in the union of $d$ $D$-balls of $M$.

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$$\Rightarrow \log(n) \text{ Rounds}$$
Lemma. Let $M$ be a set of vertices. If an open ball $B$ contains no points of $M$, then $B$ is contained in the union of $d$ $D$-balls of $M$.

Note: After $k = \log \frac{1}{\epsilon}$ rounds, the intermediate mesh is a weak $\epsilon$-net for the range space of Euclidean balls.

Size: $O\left(\frac{1}{\epsilon}\right)$, Time: $O(nk) = O(n \log \frac{1}{\epsilon})$. 

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Goes down by a factor of 2 each round.

\[\Rightarrow \log(n) \text{ Rounds}\]
To complete the analysis, we must show that the cost of a Round is $O(n)$. 

\[
\log n \text{ rounds } \times \ O(n) \text{ time/round} = O(n \log n)
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Watch an uninserted point $x$. Claim: $x$ only gets touched $O(1)$ times per round.
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$y$ touches $x$ $\Rightarrow$ $y$ is “close” to $x$

close = 2 hops among D-balls

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Only \( O(1) \) D-balls are within 2 hops.
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- $y$ touches $x \Rightarrow y$ is “close” to $x$
- close = 2 hops among D-balls
- Only $O(1)$ D-balls are within 2 hops.

Only $O(1)$ points are added to any D-ball in a round.
To complete the analysis, we must show that the cost of a Round is $O(n)$.

$$\log n \text{ rounds} \times O(n) \text{ time/round} = O(n \log n)$$

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Close = 2 hops among D-balls

Only $O(1)$ D-balls are within 2 hops.

Only $O(1)$ points are added to any D-ball in a round.

$O(n)$ total work per round.
Meshing Points in (optimal) $O(n \log n + m)$ time.

Thank you.